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Colour algebras: Lévy-Leblond equation and representation theory

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Abstract

Lie colour algebras are a generalisation of Lie superalgebras to grading by any abelian group. They have recently been found to have physical applications. For instance, it was shown in 2016 that the Lévy-Leblond equation has $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie colour algebra symmetries. In this thesis, we will examine the structure of these Lie colour algebra symmetries; in particular, we apply a technique called discolouration (which relates the structure of a Lie colour algebra to that of a Lie superalgebra) to classify these $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie colour algebra symmetries. In addition, we search for more Lie colour algebra symmetries of the Lévy-Leblond equation, and find a new $\mathbb{Z}_2^3$-graded Lie colour algebra. We show that this Lie colour algebra is fundamental to the Lévy-Leblond equation, and contains enough information to solve the equation.

Given that Lie colour algebras are finding applications, it is expected that their representation theory will be a useful tool. We show how all the finite-dimensional irreducible representations for Lie colour algebras can be found using the finite-dimensional irreducible representations for the discoloured Lie superalgebra. We strengthen this result in the case of grading by the group $\mathbb{Z}_2^n$ to obtain a bijection between these classes of representations (up to weak notions of equivalence). Despite the powerful tool of discolouration, this process is surprisingly complicated and yields an interestingly distinct representation theory for Lie colour algebras.

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Chapter 1

Introduction

Lie colour algebras are a generalisation of Lie superalgebras to grading by any abelian group. They were introduced by Rittenberg and Wyler [1, 2] in 1978, though similar structures had previously appeared in [3]. Recently, there has been renewed interest in Lie colour algebras (especially those graded by $\mathbb{Z}_2 \times \mathbb{Z}_2$) and their potential physical applications. Of particular interest to the present discussion is the appearance of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie colour algebra as a symmetry algebra of the Lévy–Leblond equation [4, 5].

The main purpose of this thesis is to examine the appearance of Lie colour algebras in physical systems and their mathematical structure. Discolouration techniques (see [6]) have rarely been applied in recent literature, despite awareness of these results. We will show how discolouration can be applied to classify the structure of the symmetry algebras in [4, 5]. We also derive a new symmetry algebra for the Lévy–Leblond equation, which has a $\mathbb{Z}_2^3$-graded structure. We show how this $\mathbb{Z}_2^3$-graded algebra can be used to aid in solving the Lévy–Leblond equation.

The representation theory of Lie colour algebras is important to their applications. With this in mind, classification results for Lie colour algebra representations would be useful. Such a desire for a classification is expressed in [7]. In this thesis, we extend the discolouration/recolouration techniques of [6] and provide a procedure to obtain the irreducible representations of Lie colour algebras from the irreducible representations of the corresponding Lie superalgebras. This procedure converts a $\mathbb{Z}_2$-graded Lie superalgebra representation to a representation graded by a larger abelian group and then applies recolouration. In the case of a $\mathbb{Z}_2^n$-grading, this procedure is a bijection between the irreducible representations for colour algebras and superalgebras (up to weak notions of equivalence). This procedure will aid in obtaining classification results.

1.1 Physical applications

After their introduction [1–3], Lie colour algebras occasionally found various applications in (for example): de Sitter spaces [8–10], quasispin [11], strings [12], extensions of Poincaré algebras [13,14], double field theories [15] and mixed tensors [16,17]. Despite these initial works, applications of colour algebras remained limited until recently.

In 2016, it was discovered that two of the symmetry algebras of the free Lévy-Leblond equation in $(1+d)$-dimensions for $d = 1, 2, 3$ were $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie colour algebras [4,5]. Lévy-Leblond [18] originally obtained his equation as a 'square root' of the $(1+3)$-dimensional Schrödinger equation and showed that it could also be obtained as a non-relativistic limit of the Dirac equation. The Lévy-Leblond equation can be generalised as a square root of a heat equation or Schrödinger equation in arbitrary spacial dimensions. That Lie colour algebras appear in such an important equation highlights the potential physical utility of Lie colour algebras. Partly due to the work in [4,5], there has recently been increased research activity in examining the physical applications of Lie colour algebras. Such research has been focused on two main areas: parastatistics and $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded quantum mechanics.

Parastatistics was first introduced by Green in [19], and replaces the ordinary particle statistics relations with nested (anti)commutation relations. Parastatistics was connected with $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebras in [20,21] and this connection was further studied in [22–24]. Certain parastatistics relations were identified as isomorphic to a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour version of $\mathfrak{osp}$ in [25] and Fock spaces for this algebra were constructed in [26]. It was recently shown that the presence of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons and parafermions is experimentally testable [27,28].

On the other hand, a simple $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded quantum mechanical model was introduced in [29] based on a supersymmetric model in [30]. A theory of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded classical mechanics was subsequently introduced in [31] based on a Lagrangian construction, and a canonical quantization procedure was developed in [32]. The development of $\mathbb{Z}_2^n$-manifolds (see e.g. [33–37]) allowed for the study of $\mathbb{Z}_2^n$-graded supersymmetry on $\mathbb{Z}_2^n$-superspace [38–41]. Recent work includes extending these systems [42–45], examining integrability [46] and bosonisation [47].

1.2 Mathematical structure

There has been continued interest in the mathematical structure of Lie colour algebras. A useful technique is that of discolouration, which yields a bijection between the class of Lie colour algebras and the class of Lie superalgebras (without colour). This bijection preserves much of the important structure, such as subalgebras, ideals and subrepresentations. Discolouration was introduced for $\mathbb{Z}_2^n$-gradings in [1], and was soon generalised [6] to grading by any finitely generated abelian group. Discolouration was used in [6] to prove Ado's theorem for Lie colour algebras (every finite-dimensional Lie colour algebra has a faithful finite-dimensional representation).

Discolouration was expressed differently in [48], with the introduction of so-called Klein operators which lie outside the universal enveloping algebra of the colour algebra. It has since been shown that these Klein operators can be expressed as members of algebraic extensions of the universal enveloping algebras for a colour version of $\mathfrak{gl}$ [49] or for the non-coloured supersymmetric $\mathfrak{gl}(m|n)$ [50].

Various works have focused on classifying Lie colour algebras. Three-dimensional [51] and four-dimensional [52] Lie colour algebras (satisfying certain properties) have been classified. Simple $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie colour algebras were classified in [53] using discolouration.

Attention has also been paid to the representation theory of Lie colour algebras. For example, the representation theories of the colour versions of the following algebras have been examined: $\mathfrak{gl}$, $\mathfrak{sl}$, $\mathfrak{osp}$ [54, 55], the Lie algebra for the group of plane motions [56], the Heisenberg Lie algebra [57], super Schrödinger algebras [58] and a supersymmetry algebra [7]. In addition, discolouration extends to representations, and it is explicitly shown in [50] how the irreducible covariant representations of $\mathfrak{gl}(m|n)$ can be lifted to representations of the colour version of $\mathfrak{gl}$. Just like the superalgebra $\mathfrak{gl}(m|n)$, not every representation of the colour version of $\mathfrak{gl}$ will be completely reducible. However, a Lie colour algebra will have the property that every finite dimensional representation is completely reducible if its discolouration also has this property [59].

1.3 Outline of the thesis

In Chapter 2 we provide the basic definitions for Lie colour algebras, their representations and related concepts. We prove some isomorphism theorems and the Jordan–Hölder theorem for graded representations, which we need for Chapter 4. We then describe the process of discolouration.

In Chapter 3, we introduce the Lévy-Leblond equation. We use discolouration to determine that the two $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry algebras in [4] are isomorphic to $\mathfrak{osp}(1, 0|2, 0) \oplus \mathfrak{osp}(1, 0|0, 2)$ and $\mathfrak{osp}(1, 0|0, 2) \oplus \mathfrak{osp}(1, 1|2, 0)$. Inspired by the work in [4], we discover a new $\mathbb{Z}_2^3$-graded symmetry algebra for the Lévy-Leblond equation. We show that this $\mathbb{Z}_2^3$-graded Lie colour algebra is fundamental to the Lévy-Leblond equation: solutions to the Lévy-Leblond equation can be expressed in terms of simultaneous eigenstates of two of the elements of this algebra.

We show, in Chapter 4, that the finite-dimensional irreducible representations of Lie colour algebras can be derived from the finite-dimensional irreducible representations of their discolourations. Discolouration does not make this a trivial procedure because one might need to convert between ungraded and graded representations (more generally, between Γ/H- and Γ-graded representations for $H \leq \Gamma$ abelian groups). To perform this conversion, we may need to increase the dimension of the representation and then quotient out a maximal submodule. In the case of a $\mathbb{Z}_2^n$-grading, we prove a stronger result: the above procedure is a bijection between the ungraded and graded representations (up to some weak notions of equivalence). As an example, we find all the finite-dimensional irreducible representations of a colour version of $\mathfrak{sl}_2$.

In Appendix A.1, we show that the Clifford algebra for the $(1+1)$-dimensional Lévy-Leblond equation has a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading, which may provide a partial explanation for the appearance of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry.

Chapter 2

Colour Algebras

Lie colour algebras are a natural generalisation of Lie superalgebras. Recall that a Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ equipped with a bracket $[\![\cdot,\cdot]\!]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Within an associative algebra, this bracket can be realised as

$$[\![x, y]\!] = xy - (-1)^{\alpha\cdot\beta} yx$$

for $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$ where $\alpha, \beta \in \mathbb{Z}_2$. The utility of a Lie superalgebra is that it can realise both commutation and anticommutation relations.

A Lie colour algebra generalises the notion of a Lie superalgebra to grading by some abelian group Γ (instead of just $\mathbb{Z}_2$). If $\mathfrak{g}$ is a Lie colour algebra, then $\mathfrak{g}$ is a Γ-graded vector space $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$ and the bracket can be realised as

$$[\![x, y]\!] = xy - \varepsilon(\alpha, \beta) yx$$

for $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$ where $\alpha, \beta \in \Gamma$. Here, $\varepsilon\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ (where $\mathbb{F}$ is the field of scalars) is called a commutation factor and, as we will soon see, must satisfy certain properties.

Many of the important structures and theorems of Lie superalgebras can be generalised to Lie colour algebras. In this section, we show how to define the following: $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour versions of the matrix algebras $\mathfrak{gl}$, $\mathfrak{sl}$ and $\mathfrak{osp}$; homomorphisms; graded and ungraded representations; irreducible representations; and quotient representations. We also prove some isomorphism theorems for graded representations, which we use to prove the Jordan–Hölder Theorem.

We finish this chapter with an explanation of how to discolour a Lie colour algebra to obtain a non-colour graded Lie superalgebra. This discoloured Lie superalgebra retains many of the properties of the original Lie colour algebra. We also show how discolouration can be extended to graded representations.

2.1 Definitions

For the definitions and examples in this section, we follow [6]. Let $\mathbb{F}$ be a field with characteristic different from 2 and 3 and let Γ be an (additive) abelian group.

Definition 2.1.1. A *commutation factor* (or *antisymmetric bicharacter*, or *phase function*) is a map $\varepsilon\colon \Gamma \times \Gamma \to \mathbb{F}^\times$ such that

$$\begin{aligned} \varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) &= 1 \\ \varepsilon(\alpha,\beta+\gamma) &= \varepsilon(\alpha,\beta)\varepsilon(\alpha,\gamma) \\ \varepsilon(\alpha+\beta,\gamma) &= \varepsilon(\alpha,\gamma)\varepsilon(\beta,\gamma) \end{aligned}$$

for all $\alpha, \beta, \gamma \in \Gamma$.

Definition 2.1.2. Given a commutation factor ε, a *Lie colour algebra* is a vector space $\mathfrak{g}$ (over $\mathbb{F}$) endowed with a bracket $[\![\cdot,\cdot]\!]$, which satisfies

$$\begin{aligned} \mathfrak{g} &= \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma, \\ [\![\mathfrak{g}_\alpha, \mathfrak{g}_\beta]\!] &\subseteq \mathfrak{g}_{\alpha+\beta} \\ [\![x,y]\!] &= -\varepsilon(\alpha,\beta)[\![y,x]\!] \\ [\![[\![x,y]\!],z]\!] &= [\![x,[\![y,z]\!]]\!] - \varepsilon(\alpha,\beta)[\![y,[\![x,z]\!]]\!] \end{aligned}$$

for $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta, z \in \mathfrak{g}_\gamma$ where $\alpha, \beta, \gamma \in \Gamma$. Elements of $\mathfrak{g}_\alpha$ are called *homogeneous elements (of degree α)*.

Example 2.1.3. If $A = \bigoplus_{\alpha\in\Gamma} A_\alpha$ is an associative algebra such that $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$ then A can be given the structure of a Lie colour algebra with bracket

$$[\![x,y]\!] = xy - \varepsilon(\alpha,\beta)yx, \qquad x \in A_\alpha,\ y \in A_\beta.$$

Example 2.1.4. A *Lie superalgebra* has $\Gamma = \mathbb{Z}_2$ with commutation factor

$$\varepsilon(\alpha,\beta) = (-1)^{\alpha\cdot\beta}$$

(here, $\alpha \cdot \beta$ denotes integer multiplication of $\alpha, \beta \in \mathbb{Z}_2 = \{0,1\}$ rather than denoting the group operation). For a Lie superalgebra, the homogeneous elements of degree 0 (resp. of degree 1) are called *even* elements (resp. *odd* elements).

Example 2.1.5. Let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$. For a group element $\alpha \in \mathbb{Z}_2 \times \mathbb{Z}_2$, we will use a simplified notation in which $\alpha = \alpha_1\alpha_2$, so that $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{00, 01, 10, 11\}$. A *$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie algebra* has commutation factor

$$\varepsilon(\alpha_1\alpha_2, \beta_1\beta_2) = (-1)^{\alpha_1\cdot\beta_2 - \alpha_2\cdot\beta_1}.$$

A *$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebra* has commutation factor

$$\varepsilon(\alpha_1\alpha_2, \beta_1\beta_2) = (-1)^{\alpha_1\cdot\beta_1+\alpha_2\cdot\beta_2}$$

(cf. [2]).

Note that every Lie colour algebra $\mathfrak{g}$ has a natural $\mathbb{Z}_2$-grading, $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$ given by

$$\mathfrak{g}^0 = \bigoplus_{\gamma\in\Gamma_0} \mathfrak{g}_\gamma, \qquad \mathfrak{g}^1 = \bigoplus_{\gamma\in\Gamma_1} \mathfrak{g}_\gamma.$$

where

$$\Gamma_0 = \{\gamma \in \Gamma \mid \varepsilon(\gamma,\gamma) = 1\} \qquad \text{and} \qquad \Gamma_1 = \{\gamma \in \Gamma \mid \varepsilon(\gamma,\gamma) \neq 1\}.$$

Either $\Gamma_0 = \Gamma$ or Γ_0 is a subgroup of Γ with index 2 [6]. However, despite this $\mathbb{Z}_2$-grading, a Lie colour algebra is not (in general) merely a Lie superalgebra. The difference lies in when the bracket is realised as a commutator or anticommutator (if it even is realised as one of these). Comparing Examples 2.1.4 and 2.1.5 should make this distinction clear.

2.2 Matrix algebras

Concrete examples of Lie colour algebras can be realised using matrices. In this section, we will present the definitions of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded matrix algebras as they appear in [2] (though we use a different convention on the order of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded sectors).

We can give a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading to $M_n(\mathbb{C})$ (the space of $n \times n$ matrices over $\mathbb{C}$) for $n = m_1 + m_2 + n_1 + n_2$ as follows:

	m_1	m_2	n_1	n_2
m_1	00	11	01	10
m_2	11	00	10	01
n_1	01	10	00	11
n_2	10	01	11	00

where each block has dimensions indicated by the row and column labels (e.g. the top right block has dimensions $m_1 \times n_2$). The 00-sector of $M_n(\mathbb{C})$ then contains the $n \times n$ matrices whose entries are zero in every block except the ones labelled by 00 in the above matrix. The assignment of the other sectors is similar. We define a bracket $[\![\cdot,\cdot]\!]$ on $M_n(\mathbb{C})$ by

$$[\![A, B]\!] = AB - (-1)^{\alpha_1\cdot\beta_1+\alpha_2\cdot\beta_2} BA$$

where $\alpha_1\alpha_2$ and $\beta_1\beta_2$ are the grades of A and B (respectively). We call $M_n(\mathbb{C})$ with this bracket the *general linear $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebra* and denote it $\mathfrak{gl}(m_1, m_2|n_1, n_2)$. Note that the general linear Lie superalgebra (with $\mathbb{Z}_2 \cong \{00, 01\}$-grading) is $\mathfrak{gl}(m|n) = \mathfrak{gl}(m, 0|n, 0)$.

For a matrix

$$A = \left[\begin{array}{c|c|c|c} A_{11} & A_{12} & A_{13} & A_{14} \\ \hline A_{21} & A_{22} & A_{23} & A_{24} \\ \hline A_{31} & A_{32} & A_{33} & A_{34} \\ \hline A_{41} & A_{42} & A_{43} & A_{44} \end{array}\right] \in \mathfrak{gl}(m_1, m_2|n_1, n_2), \tag{2.1}$$

we define its *super trace* to be

$$\operatorname{str} A = \operatorname{tr} A_{11} + \operatorname{tr} A_{22} - \operatorname{tr} A_{33} - \operatorname{tr} A_{44}.$$

The *special linear $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebra*, denoted $\mathfrak{sl}(m_1, m_2|n_1, n_2)$, contains the matrices with supertrace 0,

$$\mathfrak{sl}(m_1, m_2|n_1, n_2) = \{A \in \mathfrak{gl}(m_1, m_2|n_1, n_2) \mid \operatorname{str} A = 0\}.$$

The special linear Lie superalgebra (with $\mathbb{Z}_2 \cong \{00, 01\}$-grading) is $\mathfrak{sl}(m|n) = \mathfrak{sl}(m, 0|n, 0)$.

Let $A \in \mathfrak{gl}(m_1, m_2|n_1, n_2)$ as in (2.1). We define its *colour transpose* to be

$$A^{\mathrm{cT}} = \left[\begin{array}{c|c|c|c} A_{11}^{\mathrm{T}} & \xi\eta\zeta A_{21}^{\mathrm{T}} & -\zeta A_{31}^{\mathrm{T}} & \xi\zeta A_{41}^{\mathrm{T}} \\ \hline \xi\eta\zeta A_{12}^{\mathrm{T}} & A_{22}^{\mathrm{T}} & \xi\eta A_{32}^{\mathrm{T}} & -\eta A_{42}^{\mathrm{T}} \\ \hline \zeta A_{13}^{\mathrm{T}} & -\xi\eta A_{23}^{\mathrm{T}} & A_{33}^{\mathrm{T}} & \xi A_{43}^{\mathrm{T}} \\ \hline -\xi\zeta A_{14}^{\mathrm{T}} & \eta A_{24}^{\mathrm{T}} & \xi A_{34}^{\mathrm{T}} & A_{44}^{\mathrm{T}} \end{array}\right]$$

for a choice of numbers ξ, η, ζ such that $\xi^2 = \eta^2 = \zeta^2 = 1$.

Let n_1 and n_2 be even and set

$$S = \left[\begin{array}{c|c|c|c} I_{m_1} & 0 & 0 & 0 \\ \hline 0 & I_{m_2} & 0 & 0 \\ \hline 0 & 0 & \begin{matrix} 0 & I_{n_1/2} \\ -I_{n_1/2} & 0 \end{matrix} & 0 \\ \hline 0 & 0 & 0 & \begin{matrix} 0 & I_{n_1/2} \\ -I_{n_1/2} & 0 \end{matrix} \end{array}\right].$$

The *orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebra*, denoted $\mathfrak{osp}(m_1, m_2|n_1, n_2)$, is defined as

$$\mathfrak{osp}(m_1, m_2|n_1, n_2) = \{A \in \mathfrak{gl}(m_1, m_2|n_1, n_2) \mid A^{\mathrm{cT}} S + SA = 0\}.$$

If we choose $\zeta = 1$ in the definition of the colour transpose, then the orthosymplectic Lie superalgebra (with $\mathbb{Z}_2 \cong \{00, 01\}$-grading) is $\mathfrak{osp}(m|n) = \mathfrak{osp}(m, 0|n, 0)$.

2.3 Homomorphisms and representations

Homomorphisms, representations and related structures for Lie colour algebras are defined similarly to how they are defined for Lie superalgebras. In this section, we will specifically use the definitions as they appear in [6].

Let $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$ and $W = \bigoplus_{\gamma\in\Gamma} W_\gamma$ be vector spaces over $\mathbb{F}$.

Definition 2.3.1. A linear map $f\colon V \to W$ is *homogeneous of degree $\gamma \in \Gamma$* if $f(V_\alpha) \subseteq W_{\alpha+\gamma}$ for all $\alpha \in \Gamma$.

Definition 2.3.2. For Γ-graded Lie colour algebras $\mathfrak{g}_1, \mathfrak{g}_2$, a linear map $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is called a *homomorphism* if $[\![f(x), f(y)]\!] = f([\![x, y]\!])$ and f is homogeneous of degree 0. An *isomorphism* is a bijective homomorphism.

Example 2.3.3. Let $\mathrm{End}(V)_\gamma$ be the space of linear maps $V \to V$ that are of degree γ. Define a vector space $\mathrm{Endgr}(V) = \bigoplus_{\gamma\in\Gamma} \mathrm{End}(V)_\gamma$ and equip it with a colour bracket $[\![\cdot, \cdot]\!]$ defined by

$$[\![x, y]\!] = xy - \varepsilon(\alpha, \beta)yx, \qquad \text{for } x \in \mathrm{End}(V)_\alpha,\ y \in \mathrm{End}(V)_\beta$$

extending to inhomogeneous elements by linearity. Then $\mathrm{Endgr}(V)$ equipped with this bracket forms a Lie colour algebra, called the *general linear Lie colour algebra* and denoted $\mathfrak{gl}(V, \varepsilon)$ [6].

Definition 2.3.4. An *ungraded representation* (resp. *Γ-graded representation*) of a Lie colour algebra $\mathfrak{g}$ on a vector space V is a linear map (resp. homomorphism) $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V, \varepsilon)$ which satisfies $[\![\rho(x), \rho(y)]\!] = \rho([\![x, y]\!])$. Note that if ρ is a Γ-graded representation then it is of degree 0, so $\rho(\mathfrak{g}_\alpha)V_\xi \subseteq V_{\alpha+\xi}$. We say that V is a *$\mathfrak{g}$-module given by ρ* (or simply *a $\mathfrak{g}$-module*).

Definition 2.3.5. Let V be a $\mathfrak{g}$-module given by ρ. A *submodule* of V is a graded subspace $U = \bigoplus_{\gamma\in\Gamma} U_\gamma$ (with $U_\gamma \subseteq V_\gamma$) that satisfies $\rho(\mathfrak{g})U \subseteq U$. A non-zero $\mathfrak{g}$-module V is called *irreducible* if its only submodules are 0 and V. Otherwise, it is called *reducible*. A representation ρ is called irreducible (resp. reducible) if its corresponding $\mathfrak{g}$-module is.

Definition 2.3.6. Let V and W be $\mathfrak{g}$-modules given by ρ and π, respectively. An *intertwiner* (or *$\mathfrak{g}$-module homomorphism*) is a linear map $f\colon V \to W$ that is homogeneous of degree $0 \in \Gamma$ and $f \circ \rho(x) = \pi(x) \circ f$ for all $x \in \mathfrak{g}$. We call a bijective intertwiner a *($\mathfrak{g}$-module) isomorphism* and we say that V and W are *isomorphic*, denoted $V \cong W$, if there exists a $\mathfrak{g}$-module isomorphism from V to W.

Definition 2.3.7. Let $T(\mathfrak{g}) = \bigoplus_{j=0}^{\infty} \mathfrak{g}^{\otimes j}$ be the *tensor algebra* of $\mathfrak{g}$ and let $J(\mathfrak{g})$ be the ideal generated by elements of the form $x \otimes y - \varepsilon(\alpha, \beta)y \otimes x - [\![x, y]\!]$ for $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$. The *universal enveloping algebra* of $\mathfrak{g}$ is $U(\mathfrak{g}) = T(\mathfrak{g})/J(\mathfrak{g})$. It is standard to omit the tensor product and coset notation. For example, we write $x \otimes y + J(\mathfrak{g})$ as xy.

2.4 Graded quotient modules and isomorphism theorems

Consider a Γ-graded Lie colour algebra $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$. Let $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$ be a $\mathfrak{g}$-module given by ρ and $U = \bigoplus_{\gamma\in\Gamma} U_\gamma$ a submodule of V (so that $U_\gamma \subseteq V_\gamma$).

Definition 2.4.1. The *graded quotient module of V by U*, denoted V/U, is a direct sum of quotient vector spaces $\bigoplus_{\gamma\in\Gamma} V_\gamma/U_\gamma$ with $\mathfrak{g}$-module structure given by $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V/U, \varepsilon)$ defined by

$$\pi(x)(v + U_\xi) = \rho(x)v + U_{\alpha+\xi} \qquad \text{for } x \in \mathfrak{g}_\alpha,\ v \in V_\xi.$$

Proposition 2.4.2. *The above definition of π is a well-defined representation.*

Proof. Suppose $v_1 + U_\xi = v_2 + U_\xi$. Then $v_1 - v_2 \in U_\xi$, so $\rho(x)(v_1 - v_2) \in U_{\alpha+\xi}$ and hence $\rho(x)v_1 + U_{\alpha+\xi} = \rho(x)v_2 + U_{\alpha+\xi}$, showing that π is well-defined.

That π is a linear map $\mathfrak{g} \to \mathrm{End}(V/U)$ for which $[\![\pi(x), \pi(y)]\!] = \pi([\![x, y]\!])$ follows from the fact that ρ is a linear map for which $[\![\rho(x), \rho(y)]\!] = \rho([\![x, y]\!])$. Moreover, $\pi(\mathfrak{g}_\alpha)(V_\xi/U_\xi) \subseteq V_{\alpha+\xi}/U_{\alpha+\xi}$ by definition, so π is a representation $\mathfrak{g} \to \mathfrak{gl}(V/U, \varepsilon)$. □

Definition 2.4.3. We call U a *maximal submodule* of V if there does not exist a submodule U' of V such that $U' \neq U$, $U' \neq V$ and $U \subset U' \subset V$.

For our purposes, we are often interested in submodules U for which V/U is irreducible. In fact, V/U is irreducible if and only if U is a maximal submodule of V.

Facts which are supremely useful in the study of algebra are the isomorphism theorems. These isomorphism theorems also hold for $\mathfrak{g}$-modules. We only require two of the three isomorphism theorems (and the Butterfly Lemma)

Theorem 2.4.4 (First Isomorphism Theorem)**.** *Let $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$ and $W = \bigoplus_{\gamma\in\Gamma} W_\gamma$ be $\mathfrak{g}$-modules given by representations ρ and π, respectively. Let $f\colon V \to W$ be an intertwiner. Then,* $\ker f$ *is a submodule of V,* $\mathrm{im}\, f$ *is a submodule of W and* $\mathrm{im}\, f \cong V/\ker f$

Proof. We know that $\ker f$ is a vector space. Let $(\ker f)_\gamma = (\ker f) \cap V_\gamma$ for all $\gamma \in \Gamma$. Then $\ker f = \bigoplus_{\gamma\in\Gamma} (\ker f)_\gamma$. Let $k \in \ker f$ and $x \in \mathfrak{g}$. Since f is an intertwiner, $f(\rho(x)k) = \pi(x)f(k) = 0$, so $\rho(x)k \in \ker f$. Thus, $\ker f$ is a submodule of V.

We know that $\mathrm{im}\, f$ is a vector space. Let $(\mathrm{im}\, f)_\gamma = (\mathrm{im}\, f) \cap W_\gamma$ for all $\gamma \in \Gamma$. Then $\mathrm{im}\, f = \bigoplus_{\gamma\in\Gamma} (\mathrm{im}\, f)_\gamma$. Let $w \in \mathrm{im}\, f$ and $x \in \mathfrak{g}$. Then $w = f(v)$ for some $v \in V$. Since f is an intertwiner, $\pi(x)w = (\pi(x))f(v) = f(\rho(x)v) \in \mathrm{im}\, f$. Thus, $\mathrm{im}\, f$ is a submodule of W.

Now, define a map $\Phi\colon V/\ker f \to \mathrm{im}\, f$ by

$$\Phi(v + (\ker f)_\xi) = f(v)$$

for $v \in V_\xi$ and extending to inhomogeneous elements by linearity. We claim that Φ is a $\mathfrak{g}$-module isomorphism. First, the map Φ is well-defined; indeed, if $v_1 + (\ker f)_\xi = v_2 + (\ker f)_\xi$ then $v_1 - v_2 \in (\ker f)_\xi$ which implies $f(v_1 - v_2) = 0$ and hence $f(v_1) = f(v_2)$. Repeating the preceding argument in reverse proves injectivity. Surjectivity is obvious, and linearity follows immediately from the linearity of f. The map Φ is homogeneous of degree 0 because f is. Finally, Φ is an intertwiner: for $v \in V_\xi$ and $x \in \mathfrak{g}_\alpha$,

$$\begin{aligned}\Phi(\rho(x)v + (\ker f)_{\alpha+\xi}) &= f(\rho(x)v)\\ &= \pi(x)f(v)\\ &= \pi(x)\Phi(v + (\ker f)_\xi).\end{aligned}$$

□

Theorem 2.4.5 (Second Isomorphism Theorem)**.** *If $U = \bigoplus_{\gamma\in\Gamma} U_\gamma$ and $U' = \bigoplus_{\gamma\in\Gamma} U'_\gamma$ are submodules of a $\mathfrak{g}$-module $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$, then*

$$U'/(U' \cap U) \cong (U' + U)/U.$$

Proof. It is not difficult to verify that $U' \cap U$ and $U' + U$ are both submodules of V. Let $f : V \to V/U$ be a function defined by $f(v) = v + U_\xi$ for $v \in V_\xi$ and extending to inhomogeneous elements by linearity. Observe that $\ker f = U$. Now consider the restriction $f|_{U'}$. We have that $\ker f|_{U'} = U \cap U'$ and

$$\operatorname{im} f|_{U'} = \bigoplus_{\gamma\in\Gamma}\{u' + U_\gamma \mid u' \in U'_\gamma\} = \bigoplus_{\gamma\in\Gamma}\{u' + u + U_\gamma \mid u' \in U'_\gamma, u \in U_\gamma\} = (U' + U)/U.$$

Applying the First Isomorphism Theorem completes the proof. □

Lemma 2.4.6 (Zassenhaus Butterfly Lemma)**.** *Let V be a $\mathfrak{g}$-module and U, W, U', W' submodules of V such that $U \subseteq W$ and $U' \subseteq W'$. Then,*

$$(U + (W \cap W'))/(U + (W \cap U')) \cong (U' + (W \cap W'))/(U' + (U \cap W')).$$

Proof. The Butterfly Lemma follows from the Second Isomorphism Theorem in exactly the same way as for modules over rings (see e.g. [60, Theorem 4.7]). □

2.5 Composition Series and the Jordan–Hölder Theorem

Let V be a $\mathfrak{g}$-module. If $\mathfrak{g}$ were a semisimple Lie algebra and V were finite-dimensional, then V would be the direct sum of irreducible modules (by Weyl's Theorem). This is a very useful fact for describing the structure of $\mathfrak{g}$-modules. In cases where Weyl's Theorem does not hold, we can sometimes use irreducible submodules to describe the structure of V via composition series.

Definition 2.5.1. Consider a finite chain of submodules of V:

$$V = V^0 \supseteq V^1 \supseteq \cdots \supseteq V^r = 0.$$

A *refinement* of this chain is another finite chain

$$V = U^0 \supseteq U^1 \supseteq \cdots \supseteq U^s = 0$$

such that the multiset $\{V^0, V^1, \ldots, V^r\}$ is contained in the multiset $\{U^0, U^1, \ldots, U^r\}$.

Definition 2.5.2. Two finite chains

$$V = V^0 \supseteq V^1 \supseteq \cdots \supseteq V^r = 0. \qquad \textit{and} \qquad V = U^0 \supseteq U^1 \supseteq \cdots \supseteq U^s = 0$$

are *equivalent* if $r = s$ and there exists a permutation p on $\{0, \ldots, r-1\}$ such that $V^j/V^{j+1} \cong U^{p(j)}/U^{p(j)+1}$.

Theorem 2.5.3 (Schreier Refinement Theorem)**.** *Let V be a $\mathfrak{g}$-module. For any two finite chains of submodules, there exists a refinement of each such that the two refinements are equivalent.*

Definition 2.5.4. A *composition series* for V is a finite chain of submodules

$$V = V^0 \supseteq V^1 \supseteq \cdots \supseteq V^r = 0$$

such that V^i/V^{i+1} is irreducible for each $i = 0, \ldots, r-1$. The quotients V^i/V^{i+1} are called *composition factors*.

Theorem 2.5.5 (Jordan–Hölder)**.** *Let V be a $\mathfrak{g}$-module. Any two composition series for V are equivalent.*

The proof of the Schreier Refinement Theorem follows from the Butterfly Lemma in exactly the same way as for modules over rings (see e.g. [60, Theorem 5.7]). The Jordan–Hölder Theorem follows from the Schreier Refinement theorem [60, Theorem 5.8].

2.6 Discolouring Lie colour algebras

Scheunert [6] gave a bijection between Lie colour algebras and graded Lie superalgebras and their representations (this bijection is said to *discolour* the Lie colour algebra and their representations). This bijection has many nice properties; for example, it preserves subalgebras, ideals and subrepresentations. To compute the image of a colour algebra under this bijection, we deform the commutation factor using a multiplier.

Definition 2.6.1. A *multiplier* (or *multiplicative 2-cocycle*) is a map $\sigma\colon \Gamma \times \Gamma \to \mathbb{F}^\times$ that satisfies

$$\sigma(\alpha, \beta+\gamma)\sigma(\beta,\gamma) = \sigma(\alpha,\beta)\sigma(\alpha+\beta,\gamma).$$

for all $\alpha, \beta, \gamma \in \Gamma$.

Given a Lie colour algebra $\mathfrak{g}$ (with commutation factor ε) and a multiplier σ, we can define a new bracket $[\![\cdot,\cdot]\!]^\sigma$ by

$$[\![x,y]\!]^\sigma = \sigma(\alpha,\beta)[\![x,y]\!], \qquad x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta.$$

The vector space $\mathfrak{g}$ equipped with the bracket $[\![\cdot,\cdot]\!]^\sigma$ forms a Lie colour algebra with commutation factor ε_0 given by

$$\varepsilon_0(\alpha,\beta) = \varepsilon(\alpha,\beta)\frac{\sigma(\alpha,\beta)}{\sigma(\beta,\alpha)} \qquad \alpha,\beta \in \Gamma \tag{2.2}$$

(see [6]). We denote this Lie colour algebra by $\mathfrak{g}^\sigma$ and (following [59]) we call $\mathfrak{g}^\sigma$ the *cocycle twist* of $\mathfrak{g}$.

Assume that Γ is finitely generated. Then for any Lie colour algebra $\mathfrak{g}$ with commutation factor ε, there exists [6] a multiplier σ such that, for ε_0 given by (2.2),

$$\varepsilon_0(\alpha,\beta) = \begin{cases} -1 & \text{if } \varepsilon(\alpha,\alpha)\neq 1 \text{ and } \varepsilon(\beta,\beta)\neq 1 \\ 1 & \text{otherwise.} \end{cases} \tag{2.3}$$

In particular, $\mathfrak{g}^\sigma$ is a Lie superalgebra with even and odd sectors given by

$$\mathfrak{g}^\sigma_0 = \bigoplus_{\substack{\alpha\in\Gamma \\ \varepsilon(\alpha,\alpha)=1}} \mathfrak{g}_\alpha, \qquad \mathfrak{g}^\sigma_1 = \bigoplus_{\substack{\alpha\in\Gamma \\ \varepsilon(\alpha,\alpha)\neq 1}} \mathfrak{g}_\alpha.$$

Given the above multiplier σ, the map $\mathfrak{g}\mapsto\mathfrak{g}^\sigma$ is a bijection between the Lie colour algebras with commutation factor ε and the Γ-graded Lie superalgebras [6]. As such, we call $\mathfrak{g}^\sigma$ a *discolouration* of $\mathfrak{g}$. Discolouration is an invertible procedure, and we will call $(\mathfrak{g}^\sigma)^{1/\sigma} = \mathfrak{g}$ the *recolouration* of $\mathfrak{g}^\sigma$.

Example 2.6.2. Let $\mathfrak{g}$ be a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded colour Lie superalgebra. Define $\sigma\colon(\mathbb{Z}_2\times\mathbb{Z}_2)\times(\mathbb{Z}_2\times\mathbb{Z}_2)\to\mathbb{F}^\times$ by

$$\sigma(\alpha,\beta) = (-1)^{\alpha_1\beta_2}$$

where $\alpha=\alpha_1\alpha_2, \beta=\beta_1\beta_2\in\mathbb{Z}_2\times\mathbb{Z}_2$. We claim that σ is a multiplier; indeed,

$$\sigma(\alpha,\beta+\gamma)\sigma(\beta,\gamma) = (-1)^{\alpha_1(\beta_2+\gamma_2)+\beta_1\gamma_2} = (-1)^{\alpha_1\beta_2+(\alpha_1+\beta_1)\gamma_2} = \sigma(\alpha,\beta)\sigma(\alpha+\beta,\gamma).$$

Using (2.2), we find that

$$\begin{aligned} \varepsilon_0(\alpha,\beta) &= (-1)^{\alpha_1\beta_1+\alpha_2\beta_2+\alpha_1\beta_2-\alpha_2\beta_1} \\ &= (-1)^{\alpha_1\beta_1+\alpha_2\beta_2+\alpha_1\beta_2-\alpha_2\beta_1}(-1)^{2\alpha_2\beta_1} \\ &= (-1)^{(\alpha_1+\alpha_2)(\beta_1+\beta_2)}. \end{aligned} \tag{2.4}$$

Observe that $\varepsilon(\alpha,\alpha) = (-1)^{\alpha_1\alpha_1+\alpha_2\alpha_2} = (-1)^{\alpha_1+\alpha_2}$. So $\varepsilon(\alpha,\alpha)\neq 1$ if and only if $\alpha_1+\alpha_2=1$ (mod 2). That is, the expression for ε_0 in (2.4) agrees with (2.3). Moreover, $\alpha_1+\alpha_2$ (mod 2) corresponds to the $\mathbb{Z}_2$-grade of α in $\mathfrak{g}^\sigma$, i.e. if $x\in\mathfrak{g}_\alpha$ then $x\in\mathfrak{g}^\sigma_{\alpha_1+\alpha_2}$.

Note that there may be many choices for a multiplier which discolours $\mathfrak{g}$. However, if $\mathbb{F}$ is algebraically closed and σ_1,σ_2 are multipliers that give rise to the same commutation factor, then $\mathfrak{g}^{\sigma_1}$ is isomorphic to $\mathfrak{g}^{\sigma_2}$ [6].

There is an alternative method of discolouration which works by adding new elements, called Klein operators, to the universal enveloping algebra. These Klein operators are defined in such a way that they map the elements of the Lie colour algebra to elements which are isomorphic to the discoloured algebra defined above (see [48] for details). The advantage of this construction is that bracket for the discoloured algebra is the same as the original Lie colour algebra; we do not need to define the new bracket $[\![\cdot,\cdot]\!]^\sigma$.

In [49], it was shown that Klein operators can be expressed as an exponential of elements of the universal enveloping algebra of $\mathfrak{gl}(m_1,m_2|n_1,n_2)$. Note carefully that the Klein operators of [49] are elements of an algebraic extension of the universal enveloping algebra (one which allows exponentiation/infinite sums of elements) and are not members of the universal enveloping algebra itself. In [50] this construction was simplified by expressing the Klein operators in an algebraic extension of the universal enveloping algebra of $\mathfrak{gl}(m|n)$ where $m = m_1 + m_2$, $n = n_1 + n_2$.

Another advantage, then, of using Klein operators is that the discolouration procedure can be expressed in terms of the generators of one of the algebras of interest. However, for the remainder of this thesis we will use the original discolouration process of [6] outlined in this section.

2.7 Discolouring representations

Given a graded representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V, \varepsilon)$ and a multiplier $\sigma\colon \Gamma \times \Gamma \to \mathbb{F}^\times$, we can define its *cocycle twist* ρ^σ to be the graded representation of $\mathfrak{g}^\sigma$ given by

$$\rho^\sigma(x)v = \sigma(\alpha, \xi)\rho(x)v, \qquad \text{for } x \in \mathfrak{g}_\alpha, v \in V_\xi.$$

In particular, if σ is a multiplier which discolours $\mathfrak{g}$ then the representations of $\mathfrak{g}$ are in bijection with the graded representations of the corresponding discoloured Γ-graded Lie superalgebra [6]. Thus, the representation theory of Lie colour algebras can be completely derived from that of Γ-graded Lie superalgebras. The Klein operator version of discolouration similarly extends to representations [48], and an explicit example of this is shown in [50].

The $\mathbb{Z}_2$-graded irreducible representations of Lie superalgebras have been well studied. However, these results cannot be immediately applied to find irreducible Γ-graded representations. For instance, it might not be possible to find a Γ-grading for a $\mathbb{Z}_2$-graded representation. On the other hand, there might be an irreducible Γ-graded representation that becomes reducible when only considering the $\mathbb{Z}_2$-grading.

Chapter 3

Symmetries of the Lévy-Leblond equation

The Lévy-Leblond equation takes its inspiration from the Dirac equation, which describes a relativistic spin 1/2 particle. To obtain his equation, Dirac [61] took a 'square root' of the Klein–Gordon equation in order to replace the second-order time derivative with a first-order derivative. This square root required the introduction of gamma matrices (which generate Clifford algebras) and naturally led to the introduction of spin into the equation.

By using gamma matrices in a similar way, Lévy-Leblond obtained his equation as a 'square root' of the Schrödinger equation. Lévy-Leblond showed [18] that, just as the Dirac equation is invariant under the Poincaré group, his equation is invariant under the Galilei group. He also showed that his equation predicts the correct value for the magnetic moment of a spin 1/2 particle and that his equation can be obtained as the non-relativistic limit of the Dirac equation.

Remarkably, the Lévy-Leblond equation has symmetry algebras which are $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebras [4]. Despite the recent activity surrounding colour algebras, the Lévy-Leblond equation has received little attention since the first two papers [4, 5]. Similarly, despite its great potential utility, discolouration has received little use in recent papers.

As an attempt to fill these gaps in the literature, we use discolouration to classify the symmetry algebras in [4]. In doing so, we find that these algebras are isomorphic to $\mathfrak{osp}(1,0|2,0) \oplus \mathfrak{osp}(1,0|0,2)$ and $\mathfrak{osp}(1,0|0,2) \oplus \mathfrak{osp}(1,1|2,0)$.

Inspired by the work in [4], we also search for more Lie colour algebra symmetries of the $(1+1)$-dimensional Lévy-Leblond equation with free potential. We examine a time-independent version of the Lévy-Leblond equation as an eigenvalue-type problem and find five linearly independent operators which leave the eigenspaces invariant: the identity, a gamma matrix multiplied by the parity operator, the Schrödinger Hamiltonian, and two different 'square roots' of the Schrödinger Hamiltonian. These five operators close to form a $\mathbb{Z}_2^3$-graded Lie colour algebra. We show that this $\mathbb{Z}_2^3$-graded Lie colour algebra is fundamental to the Lévy-Leblond equation: solutions to the Lévy-Leblond equation can be expressed in terms of simultaneous eigenstates of the two 'square roots' of the Schrödinger Hamiltonian. We use this fact to solve the Lévy-Leblond equation in this simple case.

3.1 The Lévy-Leblond equation and gamma matrices

A (generalised) Lévy-Leblond equation is a first-order differential equation that is a square root of the heat or Schrödinger equation in $(1+d)$-dimensions [4]. Let t be the time coordinate and x_j be the j-th spacial coordinate. Throughout this chapter, we will use the notation $\mathbf{x} = (x_1, \ldots, x_d)$ and $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$.For the Lévy-Leblond equation with free potential, we want to find an operator Ω (which we shall call a *Lévy-Leblond operator*) such that

$$\Omega^2 = \lambda \partial_t + \Delta$$

where $\lambda \in \mathbb{C}$ is an arbitrary constant and $\Delta = \sum_j \partial_j^2$ is the Laplacian. If λ is a negative real number, then the partial differential equation induced by Ω^2 (i.e. $\Omega^2 \Psi(t, \mathbf{x}) = 0$) becomes the heat equation, and if $\lambda = i\beta$ for $\beta \in \mathbb{R}_{>0}$ (in particular, $\beta = 2m/\hbar$ where m is mass and $\hbar$ is the reduced Planck constant) then the partial differential equation induced by Ω^2 becomes the Schrödinger equation.

The goal of the Lévy-Leblond operator is to reduce a second-order differential equation (heat or Schrödinger equation) to a first-order equation. We will assume that Ω has the form

$$\Omega = \gamma_+ \partial_t + \gamma_- \lambda + \gamma^j \partial_j \tag{3.1}$$

where Einstein summation convention is used, and the coefficients $\gamma_+, \gamma_-, \gamma^j$ $(j = 1, \ldots, d)$ are yet to be determined. To have $\Omega^2 = \lambda \partial_t + \Delta$, the following anti-commutation relations must be satisfied:

$$\begin{aligned} \{\gamma_\pm, \gamma_\pm\} = 0, \quad \{\gamma_+, \gamma_-\} = 1, \\ \left\{\gamma_\pm, \gamma^j\right\} = 0, \quad \left\{\gamma^j, \gamma^k\right\} = 2\delta_{jk}, \end{aligned} \tag{3.2}$$

where δ_{jk} is the Kronecker delta. Similar to the Dirac equation, we find that these anti-commutation relations cannot be realised using numbers, and we must instead use gamma matrices.

Definition 3.1.1 ([62, Chapter 14]). The *Clifford algebra* $C\ell_{p,q}(\mathbb{R})$ is the freest algebra (over $\mathbb{R}$) generated by gamma matrices $\widetilde{\gamma}^j$, $(j = 0, \ldots, p+q-1)$ which satisfy $\left\{\widetilde{\gamma}^j, \widetilde{\gamma}^k\right\} = 2\eta_{jk}$ where the η_{jk} are entries of a symmetric matrix associated with a bilinear form of signature (p, q). More precisely, consider the tensor algebra $T(V)$ for $V = \operatorname{span}\{\gamma^j \mid j = 0, \ldots, p+q-1\}$ and let J be the ideal generated by elements of the form $\widetilde{\gamma}_j \widetilde{\gamma}_k + \widetilde{\gamma}_k \widetilde{\gamma}_j - 2\eta_{jk} 1$. Then $C\ell_{p,q}(\mathbb{R}) = T(V)/J$.

For simplicity, we will work in a basis such that

$$\eta_{jk} = \begin{cases} \delta_{jk} & \text{if } j < p, \\ -\delta_{jk} & \text{if } j \geq p. \end{cases}$$

The *complexification* of $C\ell_{p,q}(\mathbb{R})$ is $C\ell_{p,q}(\mathbb{R}) \otimes \mathbb{C}$.

Remark 3.1.2. For $(p,q) = (1,3)$, the symmetric bilinear form η_{jk} is the same bilinear form on Minkowski space [62, Chapter 14].

The gamma matrices in Definition 3.1.1 are used directly in the Dirac equation (we will discuss the relationship between the Dirac equation and the Lévy-Leblond equation in Section 3.2). Similar to the Dirac equation, we can view the gamma matrices of (3.2) as elements of some Clifford algebra. The most general Clifford algebra required is $C\ell_{2,d}(\mathbb{R}) \otimes \mathbb{C}$, but we will argue in Section 3.2 that $C\ell_{1,d}(\mathbb{R}) \otimes \mathbb{C}$ is more suitable when d is odd.

Note that Clifford algebras can be given the structure of a Lie colour algebra [63]. Since $C\ell_{1,1}(\mathbb{R}) \otimes \mathbb{C}$ can be given a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour algebra structure (see Appendix A.1), in some sense it is not surprising that $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry algebras appear in the (1+1)-dimensional Lévy-Leblond.

Remark 3.1.3. We could work entirely with real matrices, and realise the imaginary unit as a real matrix J which commutes with all the gamma matrices and squares to the negative identity matrix. This would allow us to realise the gamma matrices as a representation of a *real* Clifford algebra (see Definition 3.1.1). However, for simplicity, we will allow our gamma matrices to be over the complex numbers.

To actually solve the Lévy-Leblond equation, we would need to choose a specific matrix representation of the corresponding Clifford algebra (though many of the properties that we will study are representation independent). This makes the Lévy-Leblond equation a matrix differential equation. In particular, if we realise relations (3.2) using $n \times n$-dimensional matrices, the corresponding Hilbert space of the quantum system is $L^2(\mathbb{R}) \otimes \mathbb{C}^n$, whose elements we interpret as n-component vectors of functions. This interpretation provides a natural action of the Lévy-Leblond operator on the Hilbert space. If $\Psi(t,\mathbf{x})$ is a solution to the free Lévy-Leblond equation (i.e. $\Omega\Psi(t,\mathbf{x}) = 0$), then we also have $\Omega^2\Psi(t,\mathbf{x}) = 0$, so each component of $\Psi(t,\mathbf{x})$ will be a solution to the Schrödinger equation.

3.2 Non-relativistic limit of the Dirac equation

The Lévy-Leblond equation (corresponding to the Schrödinger equation) can be obtained as the non-relativistic limit of the Dirac equation. Recall that the Dirac equation is

$$(\widetilde{\gamma}^0 i\hbar\partial_t + \widetilde{\gamma}^j i\hbar\partial_j - mc)\Psi(t,\mathbf{x}) = 0$$

and is a relativistic equation describing a free spin 1/2 particle (see e.g. [62, Chapter 10]). The gamma matrices appearing in the Dirac equation satisfy $\{\widetilde{\gamma}^j, \widetilde{\gamma}^k\} = 2\eta_{jk}$, $(j,k = 0,\dots,d)$, where η_{jk} is a symmetric bilinear form with signature $(1,d)$:

$$\eta_{jk} = \begin{cases} \delta_{jk} & \text{if } j = 0, \\ -\delta_{jk} & \text{otherwise.} \end{cases}$$

We can also write a time-independent version of the Dirac equation, by replacing $i\hbar\partial_t$ with the total energy $\mathcal{E}$ (mc^2+ kinetic energy):

$$(\widetilde{\gamma}^0\mathcal{E} + \widetilde{\gamma}^j i\hbar\partial_j - mc)\Psi(\mathbf{x}) = 0. \tag{3.3}$$

From now on, we will work in natural units, $c = \hbar = 1$.

For the moment, we shall restrict ourselves to $(1+3)$-dimensions and choose the Dirac representation of the gamma matrices in 4×4 matrices:

$$\widetilde{\gamma}^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \widetilde{\gamma}^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

where σ^j are the Pauli matrices and I is the identity. In this representation, the Dirac equation becomes a coupled differential equation

$$\begin{cases} (\mathcal{E} - m)\varphi + \sigma^j i\partial_j\chi = 0 \\ (-\mathcal{E} - m)\chi - \sigma^j i\partial_j\varphi = 0 \end{cases} \tag{3.4}$$

where $\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$ is a solution to the Dirac equation. Lévy-Leblond discovered [18] that taking $\mathcal{E} = E + m$ (where E the non-relativistic kinetic energy), and assuming $E \ll m$, we can approximate (3.4) by

$$\begin{cases} E\varphi + \sigma^j i\partial_j\chi = 0 \\ -2m\chi - \sigma^j i\partial_j\varphi = 0. \end{cases} \tag{3.5}$$

If φ and χ solve (3.5), then $\Psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix}$ is a solution to the Lévy-Leblond equation, with

$$\gamma_+ = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \qquad \gamma_- = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}.$$

That is, the Lévy-Leblond equation is the non-relativistic limit of the Dirac equation.

We can generalise the above process for all odd space dimensions in a representation independent way. Consider the time-independent Dirac equation in $(1+d)$-dimensions, (3.3) and assume d is odd. Let

$$\widetilde{\gamma}^{\text{chir}} = i^{(d+3)d/2}\prod_{j=0}^{d}\widetilde{\gamma}^j.$$

Since the $\widetilde{\gamma}^j$ anticommute, we have that $\widetilde{\gamma}^j\widetilde{\gamma}^{\text{chir}} = (-1)^d\widetilde{\gamma}^{\text{chir}}\widetilde{\gamma}^j$, $(j = 0, \dots, d)$. Assuming that d is odd, we have that $\{\widetilde{\gamma}^j, \widetilde{\gamma}^{\text{chir}}\} = 0$. Additionally,

$$\widetilde{\gamma}^{\text{chir}} = i^{(d+3)d/2}(-1)^{d+(d-1)+\cdots+1}\prod_{j=0}^{d}\widetilde{\gamma}^{d-j} = i^{(d+3)d/2}(-1)^{(d+1)d/2}\prod_{j=0}^{d}\widetilde{\gamma}^{d-j}.$$

Therefore, $(\widetilde{\gamma}^{\text{chir}})^2 = (i^2)^{(d+3)d/2}(-1)^{(d+1)d/2+d}I = I$. In particular, $\widetilde{\gamma}^{\text{chir}}$ is invertible so Ψ is a solution to the time-independent Dirac equation if and only if

$$\widetilde{\gamma}^{\text{chir}}(\widetilde{\gamma}^0\mathcal{E} + \widetilde{\gamma}^j i\partial_j - m)\Psi(\mathbf{x}) = 0. \tag{3.6}$$

Take the non-relativistic limit by setting $\mathcal{E} = E + m$ and assuming $E \ll m$. We can compute

$$\begin{aligned}
0 &= \widetilde{\gamma}^{\text{chir}}(\widetilde{\gamma}^0\mathcal{E} + \widetilde{\gamma}^j i\partial_j - m)\Psi(\mathbf{x}) \\
&= (\widetilde{\gamma}^{\text{chir}}\widetilde{\gamma}^0(E+m) + \widetilde{\gamma}^{\text{chir}}\widetilde{\gamma}^j i\partial_j - \widetilde{\gamma}^{\text{chir}}m)\Psi(\mathbf{x}) \\
&= \left(\frac{1}{2}(\widetilde{\gamma}^{\text{chir}} + \widetilde{\gamma}^{\text{chir}}\widetilde{\gamma}^0)E - \frac{1}{2}(\widetilde{\gamma}^{\text{chir}} - \widetilde{\gamma}^{\text{chir}}\widetilde{\gamma}^0)(E+2m) + \widetilde{\gamma}^{\text{chir}}\widetilde{\gamma}^j i\partial_j\right)\Psi(\mathbf{x}) \\
&\approx (\gamma_+ E - \gamma_- 2m + \gamma^j i\partial_j)\Psi(\mathbf{x})
\end{aligned} \tag{3.7}$$

where

$$\gamma_\pm = \frac{1}{2}(\widetilde{\gamma}^{\text{chir}} \pm \widetilde{\gamma}^{\text{chir}}\widetilde{\gamma}^0), \qquad \gamma^j = \widetilde{\gamma}^{\text{chir}}\widetilde{\gamma}^j,\ (j = 1, \ldots, d). \tag{3.8}$$

It is easily verified that $\gamma_\pm$, γ^j satisfy the anti-commutation relations (3.2). We can now use the time independent Lévy-Leblond equation (3.7) to derive a time-dependent Lévy-Leblond equation by replacing E with $i\partial_t$:

$$(\gamma_+\partial_t + \gamma_- i2m + \gamma^j\partial_j)\Psi(t, \mathbf{x}) = 0$$

which has the desired form (3.1).

It is important to note that the realisation in (3.8) satisfies relations (such as $\gamma_+\gamma^j + \gamma_-\gamma^j = \gamma^j$) that are not satisfied by every representation of (3.2). Despite being less general, the realisation in (3.8) has the advantage that the gamma matrices can be defined in terms of a bilinear form with signature $(1, d)$, representing 1 time dimension and d spacial dimensions.

3.3 $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry algebras

In [4], the authors introduced two $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebras consisting of symmetry operators of the $1+1$-dimensional Lévy-Leblond square root of the free heat equation.

These $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebras were constructed from three different $\mathfrak{osp}(1|2)$ superalgebras (which all have a $\mathbb{Z}_2$-grading). We will quickly summarise the structure of each superalgebra. For simplicity, we will only provide the generators for two of these algebras, with the remaining basis elements defined by (anti)commutation relations.

The first $\mathfrak{osp}(1|2)$ algebra is generated by two elements which span the odd sector, $P_{+\frac{1}{2}}$, $P_{-\frac{1}{2}}$ defined as

$$P_{+\frac{1}{2}} = I\partial_1 \qquad\qquad P_{-\frac{1}{2}} = I\left(t\partial_1 - \frac{\lambda}{2}x_1\right) - \frac{1}{2}\gamma_+.$$

The even sector of this is then spanned by P_0, P_1, P_{-1}, defined by the anticommutation relations

$$P_{(a+b)/2} = \{P_{a/2}, P_{b/2}\} \qquad (a, b = \pm 1).$$

In addition to the above relations, this superalgebra also satisfies the following

$$\begin{aligned}
[P_0, P_s] &= 2s\lambda P_s \qquad \left(s = 0, \pm\frac{1}{2}, \pm 1\right) \\
[P_1, P_{-1}] &= -4\lambda P_0 \\
\left[P_{\pm 1}, P_{\mp\frac{1}{2}}\right] &= \mp 2\lambda P_{\pm\frac{1}{2}}.
\end{aligned} \tag{3.9}$$

The construction of the second $\mathfrak{osp}(1|2)$ superalgebra is similar. It is generated by two elements which span the odd sector, $\Omega_{+\frac{1}{2}}$, $\Omega_{-\frac{1}{2}}$ defined as

$$\Omega_{+\frac{1}{2}} = \Omega \qquad\qquad \Omega_{-\frac{1}{2}} = t\Omega$$

where $\Omega = \gamma_+\partial_t + \gamma_-\lambda + \gamma^1\partial_1$ is the Lévy-Leblond operator. The even sector is then spanned by $\Omega_0, \Omega_{+1}, \Omega_{-1}$ defined by the anticommutation relations

$$\Omega_{(a+b)/2} = \left\{\Omega_{a/2}, \Omega_{b/2}\right\} \qquad (a, b = \pm 1).$$

In addition to the above relations, this superalgebra also satisfies

$$\begin{aligned} [\Omega_0, \Omega_s] &= -2s\lambda\Omega_s \qquad \left(s = 0, \pm\frac{1}{2}, \pm 1\right) \\ [\Omega_1, \Omega_{-1}] &= 4\lambda\Omega_0 \\ \left[\Omega_{\pm 1}, \Omega_{\mp\frac{1}{2}}\right] &= \pm 2\lambda\Omega_{\pm\frac{1}{2}}. \end{aligned} \tag{3.10}$$

The third $\mathfrak{osp}(1|2)$ algebra has an even sector spanned by operators D, H, K and an odd sector spanned by operators $Q_{+\frac{1}{2}}, Q_{-\frac{1}{2}}$ defined by

$$\begin{aligned} H &= I\partial_t \\ D &= -\left(I\left(t\partial_1 + \frac{1}{2}x_1\partial_1 + \frac{1}{2}\right) + \frac{1}{4}\gamma^1\right) \\ K &= -\left(I\left(t^2\partial_t + tx_1\partial_1 - \frac{\lambda}{4}x_1^2 + t\right) - \frac{1}{2}\gamma_+x_1 + \frac{1}{2}\gamma^1 t\right) \\ Q_{+\frac{1}{2}} &= \frac{1}{\sqrt{\lambda}}(\gamma_+\partial_t - \lambda\gamma_-) \\ Q_{-\frac{1}{2}} &= \frac{1}{\sqrt{\lambda}}\left(\gamma_+\left(t\partial_t + x_1\partial_1 + \frac{1}{2}\right) - \gamma_-\lambda t - \gamma^1\frac{\lambda x_1}{2}\right) \end{aligned}$$

These operators satisfy the following relations

$$\begin{aligned} \left\{Q_{+\frac{1}{2}}, Q_{+\frac{1}{2}}\right\} &= -2H \quad & \left\{Q_{-\frac{1}{2}}, Q_{-\frac{1}{2}}\right\} &= 2K \\ \left\{Q_{+\frac{1}{2}}, Q_{-\frac{1}{2}}\right\} &= 2D & \left[D, Q_{\pm\frac{1}{2}}\right] &= \pm\frac{1}{2}Q_{\pm\frac{1}{2}} \\ \left[H, Q_{-\frac{1}{2}}\right] &= Q_{+\frac{1}{2}} & \left[K, Q_{+\frac{1}{2}}\right] &= Q_{-\frac{1}{2}} \\ [D, H] &= H & [D, K] &= -K \\ [H, K] &= 2D \end{aligned} \tag{3.11}$$

Note that the above (anti)commutation relations depend on the following relations for the gamma matrices:

$$(\gamma_\pm)^2 = 0, \qquad (\gamma^1)^2 = I, \qquad \gamma_\pm\gamma_\mp = \frac{1}{2}(I \pm \gamma^1), \qquad \gamma^1\gamma_\pm = \pm\gamma_\pm = -\gamma_\pm\gamma^1 \tag{3.12}$$

(see [4, Section 4]). Recalling that we are working with the equation in $(1+1)$-dimensions, these relations can be derived from the gamma matrices for the Dirac equation (3.8) (but not from the more general Lévy-Leblond relations in (3.2)).

In [4], two $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras were constructed using the above three $\mathfrak{osp}(1|2)$ algebras. In the remainder of this section, we discolour the two $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebras, classify the corresponding $\mathbb{Z}_2$-graded superalgebra, and then use this information to identify the original $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras.

3.3.1 The first $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebra

Let $\mathfrak{G}^{P,\Omega}$ be the first $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebra in [4], which is generated by the $P_s, \Omega_{s'}$ operators (for $s, s' = 0, \pm\frac{1}{2}, \pm 1$). In particular, the graded sectors are

$$\mathfrak{G}^{P,\Omega}_{00} = \mathrm{span}\{P_{\pm 1}, P_0, \Omega_{\pm 1}, \Omega_0\}, \quad \mathfrak{G}^{P,\Omega}_{01} = \mathrm{span}\{P_{\pm\frac{1}{2}}\}, \quad \mathfrak{G}^{P,\Omega}_{10} = \mathrm{span}\{\Omega_{\pm\frac{1}{2}}\}, \quad \mathfrak{G}^{P,\Omega}_{11} = \{0\}$$

and $[P_s, \Omega_{s'}] = 0$ for all $s, s' = 0, \pm\frac{1}{2}, \pm 1$. Therefore, $\mathfrak{G}^{P,\Omega} = \mathfrak{G}^P \oplus \mathfrak{G}^\Omega$, where $\mathfrak{G}^P$ has 00-sector $\mathrm{span}\{P_{\pm 1}, P_0\}$ and 01-sector $\mathrm{span}\{P_{\pm\frac{1}{2}}\}$ (with the 10- and 11-sectors 0-dimensional); and $\mathfrak{G}^\Omega$ has 00-sector $\mathrm{span}\{\Omega_{\pm 1}, \Omega_0\}$ and 10-sector $\mathrm{span}\{\Omega_{\pm\frac{1}{2}}\}$ (with the 01- and 11-sectors 0-dimensional). It is clear that $\mathfrak{G}^P \cong \mathfrak{osp}(1|2)$ and $\mathfrak{G}^\Omega \cong \mathfrak{osp}(1|2)$ (provided we choose the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading of each copy of $\mathfrak{osp}(1|2)$ correctly). More precisely, $\mathfrak{G}^{P,\Omega} = \mathfrak{osp}(1,0|2,0) \oplus \mathfrak{osp}(1,0|0,2)$.

We have just identified $\mathfrak{G}^{P,\Omega}$ without appealing to discolouration. However, for completeness, we will show that the discolouration of $\mathfrak{G}^{P,\Omega}$ is $(\mathfrak{G}^{P,\Omega})^\sigma \cong \mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ (where $\sigma(\alpha, \beta) = (-1)^{\alpha_1 \beta_2}$). To demonstrate this isomorphism, we will show that $[\![x, y]\!]^\sigma = [\![x, y]\!]$ for all $x, y \in \mathfrak{G}^{P,\Omega}$. Indeed, if $x \in \mathfrak{G}^{P,\Omega}_{11}$ then $x = 0$ so $[\![x, y]\!]^\sigma = 0 = [\![x, y]\!]$; similarly if $y \in \mathfrak{G}^{P,\Omega}_{11}$. If $x \in \mathfrak{G}^{P,\Omega}_{10}$ and $y \in \mathfrak{G}^{P,\Omega}_{01}$ then $[\![x, y]\!] = 0 = [\![x, y]\!]^\sigma$. If $x \in \mathfrak{G}^{P,\Omega}_\alpha$ and $y \in \mathfrak{G}^{P,\Omega}_\beta$ with $\alpha \neq 10, 11$ or $\beta \neq 01, 11$, then $\sigma(\alpha, \beta) = 1$, so $[\![x, y]\!]^\sigma = [\![x, y]\!]$ by definition. Therefore, $[\![x, y]\!]^\sigma = [\![x, y]\!]$ for all $x, y \in \mathfrak{G}^{P,\Omega}$. From this we can deduce that all the relations of $(\mathfrak{G}^{P,\Omega})^\sigma$ are the same as for $\mathfrak{G}^{P,\Omega}$ except we have $\{P_s, \Omega_{s'}\}$ instead of $[P_s, \Omega_{s'}]$ for $s, s' = \pm\frac{1}{2}$ (but not for $s, s' = 0, \pm 1$). This difference in the relations is due to the fact that, for $x \in \mathfrak{G}^{P,\Omega}_{01}$, $y \in \mathfrak{G}^{P,\Omega}_{10}$, the bracket $[\![x, y]\!]^\sigma$ is interpreted as an anticommutator (since x and y are both in the odd sector) whereas $[\![x, y]\!]$ is interpreted as a commutator (since $\varepsilon(01, 10) = 1$). Regardless, since $[\![P_s, \Omega_{s'}]\!]^\sigma = 0$ for all $s, s' = 0, \pm\frac{1}{2}, \pm 1$, we have that $(\mathfrak{G}^{P,\Omega})^\sigma = (\mathfrak{G}^P)^\sigma \oplus (\mathfrak{G}^\Omega)^\sigma$ where $(\mathfrak{G}^P)^\sigma = \mathrm{span}\{P_s \mid s = 0, \pm\frac{1}{2}, \pm 1\}$ and $(\mathfrak{G}^\Omega)^\sigma = \mathrm{span}\{\Omega_{s'} \mid s' = 0, \pm\frac{1}{2}, \pm 1\}$ are both equipped with the discoloured bracket $[\![\cdot, \cdot]\!]^\sigma$. The isomorphism $(\mathfrak{G}^{P,\Omega})^\sigma \cong \mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ follows immediately from the fact that the relations of $(\mathfrak{G}^P)^\sigma$ and the relations of $(\mathfrak{G}^\Omega)^\sigma$ both remain unchanged by discolouration, and hence both are isomorphic to $\mathfrak{osp}(1|2)$.

3.3.2 Discolouring the second $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebra

Let $\mathfrak{G}$ be the second $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebra in [4], which is generated by the $H, D, K, Q_{\pm\frac{1}{2}}, P_s$ ($s = 0, \pm\frac{1}{2}, \pm 1$) operators. The graded sectors are

$$\mathfrak{G}_{00} = \mathrm{span}\{H, D, K, P_{\pm 1}, P_0\}, \qquad \mathfrak{G}_{01} = \{P_{\pm\frac{1}{2}}\},$$
$$\mathfrak{G}_{10} = \mathrm{span}\{Q_{\pm\frac{1}{2}}, X_{\pm\frac{1}{2}}\}, \qquad \mathfrak{G}_{11} = \mathrm{span}\{X\}$$

where the operators X, $X_{\pm\frac{1}{2}}$ are defined by

$$X = \pm\Big[P_{\pm\frac{1}{2}}, Q_{\mp\frac{1}{2}}\Big] \qquad\qquad X_{\pm\frac{1}{2}} = \Big\{X, P_{\pm\frac{1}{2}}\Big\}.$$

We investigate the structure of the corresponding superalgebra $\mathfrak{G}^\sigma$ where σ is the multiplier $\sigma(\alpha,\beta) = (-1)^{\alpha_1\beta_2}$. We have that the even sector is $\mathfrak{G}^\sigma_0 = \mathfrak{G}_{00} \oplus \mathfrak{G}_{11}$ and the odd sector is $\mathfrak{G}^\sigma_1 = \mathfrak{G}_{01} \oplus \mathfrak{G}_{10}$. The (anti)commutation relations in (3.9) and (3.11) remain unchanged in $\mathfrak{G}^\sigma$. The remaining relations in $\mathfrak{G}^\sigma$ are

$$\begin{aligned}
[D, P_s]^\sigma &= sP_s & \Big[D, X_{\pm\frac{1}{2}}\Big]^\sigma &= \pm\frac{1}{2}X_{\pm\frac{1}{2}} \\
[H, P_m]^\sigma &= (1-m)P_{m+1} & \Big[H, X_{-\frac{1}{2}}\Big]^\sigma &= X_{\frac{1}{2}} \\
[K, P_m]^\sigma &= (1+m)P_{m-1} & \Big[K, X_{+\frac{1}{2}}\Big]^\sigma &= X_{-\frac{1}{2}} \\
\Big[P_{\pm1}, Q_{\mp\frac{1}{2}}\Big]^\sigma &= \pm 2X_{\pm\frac{1}{2}} & \Big[P_{\pm1}, X_{\mp\frac{1}{2}}\Big]^\sigma &= \mp 2\lambda X_{\pm\frac{1}{2}} \\
\Big[P_0, X_{\pm\frac{1}{2}}\Big]^\sigma &= \pm\lambda X_{\pm\frac{1}{2}} & \Big[P_0, Q_{\pm\frac{1}{2}}\Big]^\sigma &= \mp X_{\pm\frac{1}{2}} \\
\Big\{P_{\pm\frac{1}{2}}, X_{\mp\frac{1}{2}}\Big\}^\sigma &= \mp\lambda X & \Big\{P_{\pm\frac{1}{2}}, Q_{\mp\frac{1}{2}}\Big\}^\sigma &= \pm X \\
\Big\{X_{\frac{1}{2}}, X_{-\frac{1}{2}}\Big\}^\sigma &= \lambda P_0 & \Big\{X_{\pm\frac{1}{2}}, X_{\pm\frac{1}{2}}\Big\}^\sigma &= \lambda P_{\pm1} \\
\Big\{Q_{\pm\frac{1}{2}}, X_{\mp\frac{1}{2}}\Big\}^\sigma &= -P_0 & \Big\{Q_{\pm\frac{1}{2}}, X_{\pm\frac{1}{2}}\Big\}^\sigma &= -P_{\pm1} \\
\Big[X, X_{\pm\frac{1}{2}}\Big]^\sigma &= \lambda P_{\pm\frac{1}{2}} & \Big[H, P_{-\frac{1}{2}}\Big]^\sigma &= P_{\frac{1}{2}} \\
\Big[K, P_{+\frac{1}{2}}\Big]^\sigma &= P_{-\frac{1}{2}} & \Big[X, Q_{\pm\frac{1}{2}}\Big]^\sigma &= -P_{\pm\frac{1}{2}} \\
\Big[X, P_{\pm\frac{1}{2}}\Big]^\sigma &= -X_{\pm\frac{1}{2}}
\end{aligned} \tag{3.13}$$

for $m = 0, \pm1$, $s = 0, \pm\frac{1}{2}, \pm1$ and λ an arbitrary nonzero real number. All other (anti)commutation relations vanish. Here, we define $[x,y]^\sigma = [\![x,y]\!]^\sigma$ when $x \in \mathfrak{G}^\sigma_0$ or $y \in \mathfrak{G}^\sigma_0$, and $\{x,y\}^\sigma = [\![x,y]\!]^\sigma$ when $x, y \in \mathfrak{G}^\sigma_1$. Note that $[\cdot,\cdot]^\sigma$ and $\{\cdot,\cdot\}^\sigma$ do not necessarily coincide with the commutator $[\cdot,\cdot]$ and anticommutator $\{\cdot,\cdot\}$.

The Cartan subalgebra of $\mathfrak{G}^\sigma$ is $\mathfrak{h} = \mathrm{span}\{D, P_0, X\}$. We can then find the corresponding root system and apply standard classification results to the Lie superalgebra $\mathfrak{G}^\sigma$. In doing so, we find that $\mathfrak{G}^\sigma \cong \mathfrak{osp}(1|2) \oplus \mathfrak{sl}(2|1)$.

Alternatively, let

$$\begin{aligned}
D_P &= D - \frac{1}{2\lambda}P_0 & H_P &= H + \frac{1}{2\lambda}P_1 \\
K_P &= K - \frac{1}{2\lambda}P_{-1} & Q_{X_{\pm\frac{1}{2}}} &= Q_{\pm\frac{1}{2}} + \frac{1}{\lambda}X_{\pm\frac{1}{2}} \\
\widetilde{P}_0 &= \frac{1}{2\lambda}P_0 & \widetilde{P}^\pm &= \mp\frac{1}{2\lambda}P_{\pm1} \\
Z &= \frac{1}{2\sqrt{-\lambda}}X & F^\pm &= \pm\frac{1}{2\sqrt{-\lambda}}P_{\pm\frac{1}{2}} \pm \frac{1}{2\lambda}X_{\pm\frac{1}{2}} \\
\overline{F}^\pm &= \frac{1}{2\sqrt{-\lambda}}P_{\pm\frac{1}{2}} - \frac{1}{2\lambda}X_{\pm\frac{1}{2}}.
\end{aligned}$$

Note that $F^{\pm}$, $\overline{F}^{\pm}$ are homogeneous elements of the odd sector, even though they are not homogeneous elements in the original $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading. (We do not need to respect the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading in this case since we are examining the discolouration.) It is easily verified that $D_P, H_P, K_P, Q_{X_{\pm\frac{1}{2}}}$ satisfy relations (3.11) under $[\cdot,\cdot]^\sigma$ and $\{\cdot,\cdot\}^\sigma$. Thus,

$$\mathrm{span}\{D_P, H_P, K_P, Q_{X_{\pm\frac{1}{2}}}\} \cong \mathfrak{osp}(1|2).$$

Additionally, the operators $Z, \widetilde{P}_0, \widetilde{P}^{\pm}, F^{\pm}, \overline{F}^{\pm}$ satisfy

$$\begin{aligned}
\left[\widetilde{P}_0, \widetilde{P}^{\pm}\right]^\sigma &= \pm\widetilde{P}^{\pm} & \left[\widetilde{P}^{+}, \widetilde{P}^{-}\right]^\sigma &= 2\widetilde{P}_0\\
\left[Z, \widetilde{P}^{\pm}\right]^\sigma &= \left[Z, \widetilde{P}_0\right]^\sigma = 0 & \left[\widetilde{P}_0, F^{\pm}\right]^\sigma &= \pm\frac{1}{2}F^{\pm}\\
\left[\widetilde{P}_0, \overline{F}^{\pm}\right]^\sigma &= \pm\frac{1}{2}\overline{F}^{\pm} & \left[\widetilde{P}^{\pm}, F^{\mp}\right]^\sigma &= -F^{\pm}\\
\left[\widetilde{P}^{\pm}, \overline{F}^{\mp}\right]^\sigma &= \overline{F}^{\pm} & \left[\widetilde{P}^{\pm}, F^{\pm}\right]^\sigma &= \left[\widetilde{P}^{\pm}, \overline{F}^{\pm}\right]^\sigma = 0\\
\left[Z, F^{\pm}\right]^\sigma &= \frac{1}{2}F^{\pm} & \left[Z, \overline{F}^{\pm}\right]^\sigma &= -\frac{1}{2}\overline{F}^{\pm}\\
\left\{F^{\pm}, F^{\pm}\right\}^\sigma &= \left\{F^{\pm}, F^{\mp}\right\}^\sigma = 0 & \left\{\overline{F}^{\pm}, \overline{F}^{\pm}\right\}^\sigma &= \left\{\overline{F}^{\pm}, \overline{F}^{\mp}\right\}^\sigma = 0\\
\left\{F^{\pm}, \overline{F}^{\pm}\right\}^\sigma &= \widetilde{P}^{\pm} & \left\{F^{\pm}, \overline{F}^{\mp}\right\}^\sigma &= Z \mp \widetilde{P}_0
\end{aligned}$$

which are commutation relations for $\mathfrak{sl}(2|1)$. One can also check that the above generators for $\mathfrak{osp}(1|2)$ (anti)commute with the generators for $\mathfrak{sl}(2|1)$, showing that $\mathfrak{G}^\sigma \cong \mathfrak{osp}(1|2) \oplus \mathfrak{sl}(2|1)$.

3.3.3 Recolouring the second $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebra

We can use the structure of $\mathfrak{G}^\sigma$ to classify $\mathfrak{G}$. As in Section 3.3.1, we can argue that the $\mathfrak{osp}(1|2)$ summand is unchanged by discolouration. Therefore, $\mathfrak{G}$ contains $\mathfrak{osp}(1|2)$ as a summand (more precisely, a summand of $\mathfrak{osp}(1,0|0,2)$). It is known that $\mathfrak{sl}(2|1) \cong \mathfrak{osp}(2|2)$, and that $\mathfrak{osp}(2|2)$ is the discolouration of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie colour algebra $\mathfrak{osp}(1,1|2,0)$ [64]. It turns out that $\mathfrak{G} \cong \mathfrak{osp}(1,0|0,2) \oplus \mathfrak{osp}(1,1|2,0)$.

Recall [65] that $\mathfrak{osp}(2|2)$ can be realised as matrices of the form

$$\left[\begin{array}{cc|cc} 0 & -a & x_1 & y_1\\ a & 0 & x_2 & y_2\\ \hline -y_1 & -y_2 & b & c\\ x_1 & x_2 & d & -b \end{array}\right] \quad \text{with grading} \quad \left[\begin{array}{c|c} 0 & 1\\ \hline 1 & 0 \end{array}\right]$$

for $a, b, c, d, x_1, x_2, y_1, y_2 \in \mathbb{C}$. The bracket on $\mathfrak{osp}(2|2)$ is given by

$$[\![A, B]\!] = AB - (-1)^{\alpha\cdot\beta} BA$$

where α, β are the $\mathbb{Z}_2$-grade of A, B (respectively).

For the space of 4×4 matrices, define a basis $\{E_{IJ} \mid I, J = 1, \dots, 4\}$ by $(E_{IJ})_{ij} = \delta_{Ii}\delta_{Jj}$. If we set

$$\begin{aligned} P &= \lambda(E_{4,4} - E_{3,3}) & P_{+1} &= 2\lambda E_{4,3} & P_{-1} &= 2\lambda E_{3,4} \\ P_{+\frac{1}{2}} &= \sqrt{\lambda}\,(E_{1,3} + E_{4,1}) & P_{-\frac{1}{2}} &= \sqrt{\lambda}\,(E_{1,4} - E_{3,1}) & X &= \sqrt{\lambda}\,(E_{1,2} - E_{2,1}) \\ X_{+\frac{1}{2}} &= \lambda(E_{2,3} + E_{4,2}) & X_{-\frac{1}{2}} &= \lambda(E_{2,4} - E_{3,2}) \end{aligned}$$

then $\{P, P_{\pm 1}, P_{\pm\frac{1}{2}}, X, X_{\pm\frac{1}{2}}\}$ span $\mathfrak{osp}(2|2)$ and satisfy relations (3.9) and (3.13) under the (anti)commutator. (However, note that the above matrix realisation does not act on the original Hilbert space containing states of the quantum system.)

The above discussion provides an explicit isomorphism between the $\mathfrak{sl}(2|1)$ factor of $\mathfrak{G}^{\sigma}$ and $\mathfrak{osp}(2|2)$. Next, we can recolour $\mathfrak{osp}(2|2)$.

The $\mathfrak{osp}(2|2)$ algebra can be turned into a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra as follows:

$$\left[\begin{array}{c|c|cc} 0 & -a & x_1 & y_1 \\ \hline a & 0 & x_2 & y_2 \\ \hline -y_1 & -y_2 & b & c \\ x_1 & x_2 & d & -b \end{array}\right] \quad \text{with grading} \quad \left[\begin{array}{c|c|c} 00 & 11 & 01 \\ \hline 11 & 00 & 10 \\ \hline 01 & 10 & 00 \end{array}\right].$$

If we interpret the matrices of $\mathfrak{osp}(2|2)$ as a representation acting on a four dimensional representation space, whose column vectors have the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading

$$\left[\begin{array}{c} 00 \\ \hline 11 \\ \hline 01 \\ 01 \end{array}\right],$$

then we can use $\sigma'(\alpha, \beta) = (-1)^{\alpha_2 \cdot \beta_1}$ to obtain a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebra. The 2-cocycle σ' multiplies each block of the matrices in $\mathfrak{osp}(2|2)$ by either $+1$ or -1 as per

$$\left[\begin{array}{c|c|c} +1 & -1 & +1 \\ \hline +1 & +1 & +1 \\ \hline +1 & +1 & +1 \end{array}\right].$$

The result forms a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebra with bracket

$$[\![A, B]\!] = AB - (-1)^{\alpha_1 \cdot \beta_1 + \alpha_2 \cdot \beta_2} BA$$

where $\alpha_1\alpha_2$ and $\beta_1\beta_2$ are the grades of A and B (respectively). This colour superalgebra is $\mathfrak{osp}(1,1|2,0)$. We have thus shown that $\mathfrak{G} \cong \mathfrak{osp}(1,0|0,2) \oplus \mathfrak{osp}(1,1|2,0)$.

Remark 3.3.1. If we choose $\xi = \eta = 1$ and $\zeta = -1$ in the definition of colour transpose, then we can easily verify that $\mathfrak{osp}(1,1|2,0)$ as defined in Section 2.2 is the same as the $\mathfrak{osp}(1,1|2,0)$ used above.

3.4 Symmetries of the free Lévy-Leblond equation

Inspired by the symmetry algebras in [4] (see also, Section 3.3), in this section, we wish to identify the symmetry operators which leave the eigenspaces invariant, and determine if any $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry appears.

3.4.1 Initial observations

Recall that the $(1+1)$-dimensional Lévy-Leblond equation (for the Schrödinger equation) with free potential is

$$(\gamma_+ i\partial_t - \gamma_-\beta + \gamma^1 i\partial_1)\Psi(t, x_1) = 0 \tag{3.14}$$

where t is the time coordinate and x_1 is the first (and only) spacial coordinate. We can choose to interpret $\gamma_-\beta - \gamma^1 i\partial_1$ as a pseudo-Hamiltonian and set up a time-independent version of this equation:

$$(\gamma_-\beta - \gamma^1 i\partial_1)\psi(x_1) = \gamma_+ E\psi(x_1) \tag{3.15}$$

for some $E \in \mathbb{C}$. A difficulty that we encounter with the Lévy-Leblond equation (which we would not encounter while solving the Dirac equation) is that $(\gamma_+)^2 = 0$ so γ_+ is not invertible. This means that we are forced to leave γ_+ on the right-hand side of (3.15).

If $\psi(x_1)$ is a solution to the time-independent equation (3.15), then

$$\Psi(t, x_1) = e^{-iEt}\psi(x_1)$$

is a solution to the time-dependent equation (3.14).

Let $|\psi\rangle \equiv \psi(x_1)$ be a solution to the time-independent equation (3.15). By abuse of terminology, we will call $|\psi\rangle$ an eigenstate with eigenvalue $\gamma_+ E$ and use associated terminology (such as eigenspaces).

We can rearrange equation (3.15) to solve for ∂_1:

$$\begin{aligned} \partial_1 |\psi\rangle &= (\gamma^1\gamma_+ iE - \gamma^1\gamma_- i\beta)\,|\psi\rangle \\ &= (\gamma_+ iE + \gamma_- i\beta)\,|\psi\rangle \end{aligned} \tag{3.16}$$

using relations (3.12). Squaring both sides (noting that ∂_1 commutes with the operator on the right-hand side), we find that

$$(\partial_1)^2 |\psi\rangle = -E\beta\,|\psi\rangle \qquad \Longrightarrow \qquad -\frac{1}{\beta}(\partial_1)^2 |\psi\rangle = E\,|\psi\rangle\,. \tag{3.17}$$

Since $-(1/\beta)(\partial_1)^2$ is the Hamiltonian for the free Schrödinger equation, we have that $|\psi\rangle$ is an eigenstate for the Schrödinger Hamiltonian. In particular $-(1/\beta)(\partial_1)^2$ is proportional to the square of the self-adjoint momentum operator $-i\partial_1$, and hence is a self-adjoint non-negative operator. Consequently, its spectrum must be contained in $[0, \infty)$ (see e.g. [66, Proposition 9.20]). That is, $E \in [0, \infty)$.

3.4.2 Eigenvalue conditions

We wish to find an operator A that leaves the eigenspaces invariant; that is,

$$(\gamma_- \beta - \gamma^1 i\partial_1)A\,|\psi\rangle = \gamma_+ EA\,|\psi\rangle \tag{3.18}$$

for $|\psi\rangle$ a solution to the time-independent equation (3.15). Equation (3.16) shows that the operator ∂_1 can be expressed in terms of gamma matrices when acting on an eigenspace. Therefore, if we only consider the action on an eigenspace, A does not need to contain any differential operators. So the general form of A (restricted to an eigenspace) is

$$A = c_I(x_1)I + c_+(x_1)\gamma_+ + c_-(x_1)\gamma_- + c_1(x_1)\gamma^1 \tag{3.19}$$

for some complex-valued functions c_I, c_+, c_-, c_1. For simplicity, we will assume that these functions are differentiable. (The fact that this general form of A does not contain any differential operators is unique to the $(1+1)$-dimensional case.) We then substitute the general A in (3.19) into (3.18), and use relations (3.12), (3.16) and the product rule: $\partial_1 f(x_1) = f(x_1)\partial_1 + f'(x_1)$ for any differentiable function f. In doing so, we obtain a coupled ODE:

$$\begin{cases} \dfrac{\mathrm{d}c_I}{\mathrm{d}x_1} = 0 \\ \dfrac{\mathrm{d}c_+}{\mathrm{d}x_1} = -2iEc_1(x_1) \\ \dfrac{\mathrm{d}c_-}{\mathrm{d}x_1} = 2i\beta c_1(x_1) \\ \dfrac{\mathrm{d}c_1}{\mathrm{d}x_1} = -i\beta c_+(x_1) + iEc_-(x_1). \end{cases} \tag{3.20}$$

Obviously $c_I(x_1) \equiv c_I$ is a constant. Defining a new variable $c(x_1) = -i\beta c_+(x_1) + iEc_-(x_1)$, yields the following coupled ODE:

$$\begin{cases} \dfrac{\mathrm{d}c}{\mathrm{d}x_1} = -4E\beta c_1(x_1) \\ \dfrac{\mathrm{d}c_1}{\mathrm{d}x_1} = c(x_1). \end{cases}$$

which we can solve to find that

$$\begin{aligned} c_1(x_1) &= ae^{i2\sqrt{E\beta}\,x_1} + be^{-i2\sqrt{E\beta}\,x_1} \\ c(x_1) &= i2\sqrt{E\beta}\,(ae^{i2\sqrt{E\beta}\,x_1} - be^{-i2\sqrt{E\beta}\,x_1}). \end{aligned}$$

Using the ODE (3.20) and the constraint $-i\beta c_+(x_1) + iEc_-(x_1) = c(x_1)$, we can solve for c_+ and c_-:

$$\begin{aligned} c_+(x_1) &= -a\sqrt{\tfrac{E}{\beta}}\,e^{i2\sqrt{E\beta}\,x_1} + b\sqrt{\tfrac{E}{\beta}}\,e^{-i2\sqrt{E\beta}\,x_1} + \frac{d}{\beta} \\ c_-(x_1) &= a\sqrt{\tfrac{\beta}{E}}\,e^{i2\sqrt{E\beta}\,x_1} - b\sqrt{\tfrac{\beta}{E}}\,e^{-i2\sqrt{E\beta}\,x_1} + \frac{d}{E} \end{aligned}$$

for some constants a, b, d and assuming E and β are non-zero. Substituting the expressions we have computed for c_I, c_+, c_-, c_1 back into the general form of A in (3.19), we find four linearly independent operators:

$$I \tag{3.21}$$

$$\gamma_+ iE + \gamma_- i\beta \tag{3.22}$$

$$-\gamma_+\sqrt{\frac{E}{\beta}}\, e^{i2\sqrt{E\beta}\,x_1} + \gamma_-\sqrt{\frac{\beta}{E}}\, e^{i2\sqrt{E\beta}\,x_1} + \gamma^1 e^{i2\sqrt{E\beta}\,x_1} \tag{3.23}$$

$$\gamma_+\sqrt{\frac{E}{\beta}}\, e^{-i2\sqrt{E\beta}\,x_1} - \gamma_-\sqrt{\frac{\beta}{E}}\, e^{-i2\sqrt{E\beta}\,x_1} + \gamma^1 e^{-i2\sqrt{E\beta}\,x_1}. \tag{3.24}$$

We can easily verify that these operators leave the eigenspace invariant.

For a specific choice of E, the operators (3.21) – (3.24) are only defined on the corresponding eigenspace. If we want operators defined everywhere on (a dense subset of) the Hilbert space, then we can interpret (3.21) – (3.24) as eigenvalue conditions which the desired operators must satisfy. We could possibly appeal to the functional calculus from the Spectral Theorem (see e.g. [66, Definition 10.5]) to obtain operators defined everywhere. However, we are able to obtain some operators from (3.21) – (3.24) without using this functional calculus.

3.4.3 Finding operators

From (3.16) we immediately see that the operator ∂_1 has the same action as (3.22) on the eigenspace. (Note that we could have multiplied by $-i$ and instead chosen the momentum operator $-i\partial_1$, which is self-adjoint. We will not concern ourselves with self-adjointness for now.) Since ∂_1 commutes with the pseudo-Hamiltonian $H_{\mathrm{LL}} := \gamma_-\beta - \gamma^1 i\partial_1$ and with $\gamma_+ E$, we can easily verify that ∂_1 does indeed leave the eigenspaces invariant. Surprisingly, (3.22) also gives rise to another operator. If we multiply (3.22) by $-i\beta$ and use the fact, from (3.17), that E is an eigenvalue of the Schrödinger Hamiltonian, then we obtain the operator $-\gamma_+(\partial_1)^2 + \gamma_-\beta^2$. This operator does leave the eigenspaces invariant:

$$\begin{aligned} H_{\mathrm{LL}}(-\gamma_+(\partial_1)^2 + \gamma_-\beta^2)\,|\psi\rangle &= -i\beta H_{\mathrm{LL}}(\gamma_+ iE + \gamma_- i\beta)\,|\psi\rangle \\ &= -i\beta\gamma_+ E(\gamma_+ iE + \gamma_- i\beta)\,|\psi\rangle \\ &= \gamma_+ E(-\gamma_+(\partial_1)^2 + \gamma_-\beta^2)\,|\psi\rangle \end{aligned}$$

using the fact that $\gamma_+ iE + \gamma_- i\beta$ leaves the eigenspace invariant. (Note that we could have also used a similar argument for ∂_1.)

Now, we shall examine (3.24). We need to find an operator which has $|\psi\rangle$ as an eigenstate with eigenvalue involving $\sqrt{E}$. We would expect the eigenvalues of $(\partial_1)^2$ to be the squares of the eigenvalues of ∂_1; that is, we expect the eigenvalues of ∂_1 to be $\pm i\sqrt{E\beta}$. For now, we shall assume that $|\psi\rangle$ is an eigenstate of ∂_1 with eigenvalue $i\sqrt{E\beta}$. With this eigenvalue, we can find an operator corresponding to $e^{-2i\sqrt{E\beta}\,x_1}$:

$$\begin{aligned} e^{-2i\sqrt{E\beta}\,x_1}\,|\psi\rangle &= \sum_{k=0}^{\infty} \frac{(-2x_1)^k (i\sqrt{E\beta}\,)^k \psi(x_1)}{k!} \qquad\qquad \text{where } \psi(x_1) \equiv |\psi\rangle \\ &= \sum_{k=0}^{\infty} \frac{(-2x_1)^k (\partial_1)^k \psi(x_1)}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\psi^{(k)}(x_1)}{k!} (-x_1 - x_1)^k. \end{aligned}$$

This final expression is the Taylor series for $\psi(-x_1)$ about the point x_1. If we assume that ψ is analytic everywhere, then $e^{-2i\sqrt{E\beta}\,x_1}\,|\psi\rangle = \psi(-x_1)$. So, on an analytic function, $e^{-2i\sqrt{E\beta}}$ acts the same as the parity operator $\mathcal{P}$, defined by $\mathcal{P}(\chi(x_1)) = \chi(-x_1)$ for all states χ. Note that $\partial_1 \mathcal{P} = -\mathcal{P}\partial_1$ by the chain rule.

Assuming that $|\psi\rangle$ is an analytic function, then from (3.24) we compute:

$$\begin{aligned} &\left(\gamma_+ \sqrt{\frac{E}{\beta}}\, e^{-i2\sqrt{E\beta}\,x_1} - \gamma_- \sqrt{\frac{\beta}{E}}\, e^{-i2\sqrt{E\beta}\,x_1} + \gamma^1 e^{-i2\sqrt{E\beta}\,x_1}\right) |\psi\rangle \\ &= \left(\gamma_+ \frac{iE}{i\sqrt{E\beta}} \mathcal{P} - \gamma_- \frac{i\beta}{i\sqrt{E\beta}} \mathcal{P} + \gamma^1 \mathcal{P}\right) |\psi\rangle \\ &= \mathcal{P} \left(\gamma^1 \frac{1}{i\sqrt{E\beta}} \partial_1 + \gamma^1\right) |\psi\rangle \\ &= 2\gamma^1 \mathcal{P}\,|\psi\rangle \end{aligned}$$

using (3.16) and the fact that ∂_1 has eigenvalue $i\sqrt{E\beta}$. Therefore, we guess that $\gamma^1\mathcal{P}$ is an operator which leaves the eigenspace invariant. And indeed, we can verify that this is true for all eigenstates $|\psi\rangle$ (not just analytic ones):

$$\begin{aligned} H_{\mathrm{LL}}(\gamma^1 \mathcal{P}\,|\psi\rangle) &= \gamma_- \beta \mathcal{P}\,|\psi\rangle - i\partial_1 \mathcal{P}\,|\psi\rangle \\ &= \gamma_- \beta \mathcal{P}\,|\psi\rangle + i\mathcal{P}\partial_1\,|\psi\rangle \\ &= \gamma_- \beta \mathcal{P}\,|\psi\rangle + i\mathcal{P}(\gamma_+ iE + \gamma_- i\beta)\,|\psi\rangle \\ &= -\gamma_+ E \mathcal{P}\,|\psi\rangle \\ &= \gamma_+ E(\gamma^1 \mathcal{P}\,|\psi\rangle). \end{aligned}$$

Note that we did not use the assumption that the eigenvalue of ∂_1 is $i\sqrt{E\beta}$ in the above computation.

To find $\gamma^1\mathcal{P}$, we assumed that ∂_1 has eigenvalue $i\sqrt{E\beta}$. If we instead assume that ∂_1 has eigenvalue $-i\sqrt{E\beta}$, then the operator we obtain from (3.24) is 0, since

$$\begin{aligned} &\left(\gamma_+ \sqrt{\frac{E}{\beta}}\, e^{-i2\sqrt{E\beta}\,x_1} - \gamma_- \sqrt{\frac{\beta}{E}}\, e^{-i2\sqrt{E\beta}\,x_1} + \gamma^1 e^{-i2\sqrt{E\beta}\,x_1}\right) |\psi\rangle \\ &= e^{-i2\sqrt{E\beta}\,x_1} \left(\gamma_1 \frac{1}{i\sqrt{E\beta}} \partial_1 + \gamma^1\right) |\psi\rangle\,. \end{aligned}$$

The behaviour for (3.23) is similar: if ∂_1 has eigenvalue $-i\sqrt{E\beta}$ then we again obtain the operator $\gamma^1\mathcal{P}$ and if ∂_1 has eigenvalue $i\sqrt{E\beta}$ then we obtain the zero operator.

So far, we have identified three operators which leave the eigenspace invariant: ∂_1, $-\gamma_+(\partial_1)^2+\gamma_-\beta^2$, $\gamma^1\mathcal{P}$. Arguments identical to the ones for these operators also show that $(\partial_1)^k$, $-\gamma_+(\partial_1)^{k+2}+\gamma_-\beta^2(\partial_1)^k$, $\gamma^1\mathcal{P}(\partial_1)^k$ $(k = 1, 2, \ldots)$ also leave the eigenspace invariant. We can also obtain these operators from multiplying (3.22) – (3.24) by $(\pm i\sqrt{E\beta}\,)^{k-1}$ (the eigenvalue of $(\partial_1)^{k-1}$). Additionally, we can replace $-i\beta\partial_1$ with $-\gamma_+(\partial_1)^2 + \gamma_-\beta^2$ since both act as $\gamma_+E\beta + \gamma_-\beta^2$ on the eigenspace. Therefore,

$$\begin{gathered}
(\partial_1)^k(-\gamma_+(\partial_1)^2 + \gamma_-\beta^2)\\
(-\gamma_+(\partial_1)^{k+2} + \gamma_-\beta^2(\partial_1)^k)(-\gamma_+(\partial_1)^2 + \gamma_-\beta^2)\\
\gamma^1\mathcal{P}(\partial_1)^k(-\gamma_+(\partial_1)^2 + \gamma_-\beta^2)
\end{gathered}$$

also leave the eigenspace invariant. (Note that $(-\gamma_+(\partial_1)^2 + \gamma_-\beta^2)^2 = -(\beta\partial_1)^2$ so we do not need to consider higher powers.)

3.4.4 Grading the operators

We have an infinite number of operators which leave the eigenspaces invariant. Since infinite-dimensional objects are difficult to work with, we want to construct a finite-dimensional algebra whose universal enveloping algebra contains all of these operators.

Let,

$$H_{\mathrm{Sch}} = -\frac{1}{\beta}(\partial_1)^2, \qquad \widehat{P} = -i\partial_1, \qquad D_+ = -\gamma_+\frac{1}{\beta}(\partial_1)^2 + \gamma_-\beta, \qquad \mathcal{P}^1 = \gamma^1\mathcal{P}.$$

Computing the complete set of commutation and anticommutation relations we find

$$\begin{aligned}
[\widehat{P}, \widehat{P}] &= 0 & \{\widehat{P}, \widehat{P}\} &= 2\beta H_{\mathrm{Sch}}\\
[D_+, D_+] &= 0 & \{D_+, D_+\} &= 2\beta H_{\mathrm{Sch}}\\
[\mathcal{P}^1, \mathcal{P}^1] &= 0 & \{\mathcal{P}^1, \mathcal{P}^1\} &= 2I\\
[\widehat{P}, D_+] &= 0 & \{\widehat{P}, D_+\} &= 2D_+\widehat{P}\\
[\widehat{P}, \mathcal{P}^1] &= -2\mathcal{P}^1\widehat{P} & \{\widehat{P}, \mathcal{P}^1\} &= 0\\
[D_+, \mathcal{P}^1] &= -2\mathcal{P}^1 D_+ & \{D_+, \mathcal{P}^1\} &= 0
\end{aligned} \tag{3.25}$$

and H_{Sch} commutes with the other three operators.

We can create Lie colour algebras by assigning gradings to these operators to choose which of the above (anti)commutation relations are expressed.

Theorem 3.4.1. *Let $\mathfrak{A}$, $\mathfrak{D}^+$, $\mathfrak{D}^1$, $\mathfrak{D}$ be vector spaces. Define $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded sectors for*

$\mathfrak{A}$, $\mathfrak{D}^+$, $\mathfrak{D}^1$ *and* $\mathbb{Z}_2^3$*-graded sectors for* $\mathfrak{D}$ *as follows:*

$$\begin{array}{llll}
\mathfrak{A}_{00} = 0, & \mathfrak{A}_{01} = \mathrm{span}\{\widehat{P}, D_+\}, & \mathfrak{A}_{10} = \mathrm{span}\{\mathcal{P}^1\}, & \mathfrak{A}_{11} = 0, \\
\mathfrak{D}^+_{00} = \mathrm{span}\{I, H_{Sch}\}, & \mathfrak{D}^+_{01} = \mathrm{span}\{\widehat{P}\}, & \mathfrak{D}^+_{10} = \mathrm{span}\{D_+\}, & \mathfrak{D}^+_{11} = \mathrm{span}\{\mathcal{P}^1\}, \\
\mathfrak{D}^1_{00} = \mathrm{span}\{I\}, & \mathfrak{D}^1_{01} = \mathrm{span}\{\mathcal{P}^1\}, & \mathfrak{D}^1_{10} = 0, & \mathfrak{D}^1_{11} = \mathrm{span}\{\widehat{P}, D_+\},
\end{array}$$

$$\begin{array}{llll}
\mathfrak{D}_{000} = \mathrm{span}\{I, H_{Sch}\}, & \mathfrak{D}_{001} = \mathrm{span}\{\widehat{P}\}, & \mathfrak{D}_{010} = \mathrm{span}\{D_+\}, & \mathfrak{D}_{011} = 0, \\
\mathfrak{D}_{100} = 0, & \mathfrak{D}_{101} = 0, & \mathfrak{D}_{110} = 0, & \mathfrak{D}_{111} = \mathrm{span}\{\mathcal{P}^1\}.
\end{array}$$

We then define $\mathfrak{A}$, $\mathfrak{D}^+$, $\mathfrak{D}^1$, $\mathfrak{D}$ *to be the direct sum of their respective sectors. Define a bracket on each space by* $[\![A, B]\!] = AB - \varepsilon(\alpha, \beta)BA$ *for* A, B *homogeneous of degree* α, β *respectively. Here,* ε *is the commutation factor for the algebra*

$$\begin{array}{rr}
\textit{for } \mathfrak{A}: & \varepsilon(\alpha_1\alpha_2, \beta_1\beta_2) = (-1)^{\alpha_1\cdot\beta_2 - \alpha_2\cdot\beta_1} \\
\textit{for } \mathfrak{D}^+ \textit{ and } \mathfrak{D}^1: & \varepsilon(\alpha_1\alpha_2, \beta_1\beta_2) = (-1)^{\alpha_1\cdot\beta_1 + \alpha_2\cdot\beta_2} \\
\textit{for } \mathfrak{D}: & \varepsilon(\alpha_1\alpha_2\alpha_3, \beta_1\beta_2\beta_3) = (-1)^{\alpha_1\cdot\beta_1 + \alpha_2\cdot\beta_2 + \alpha_3\cdot\beta_3}.
\end{array}$$

Then, $\mathfrak{A}$*,* $\mathfrak{D}^+$*,* $\mathfrak{D}^-$*, and* $\mathfrak{D}$ *close to form Lie colour algebras.*

Proof. By examining (3.25), we verify that the bracket of the α-sector and the β-sector is a subspace of the $(\alpha + \beta)$-sector (for all possible sectors). The remaining properties follow immediately from the definition of $[\![\cdot, \cdot]\!]$ (cf. Example 2.1.3). □

If we want to construct an algebra that contains only $\widehat{P}$, D_+, $\mathcal{P}^1$, then we need to define a bracket $[\![\cdot, \cdot]\!]$ such that $[\![\widehat{P}, \widehat{P}]\!] = [\widehat{P}, \widehat{P}]$ but $[\![\widehat{P}, \mathcal{P}^1]\!] = \{\widehat{P}, \mathcal{P}^1\}$. This is not possible with a Lie superalgebra, but is possible with a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie algebra such as $\mathfrak{A}$. However, $\mathfrak{A}$ is somewhat trivial because $[\![A, B]\!] = 0$ for all $A, B \in \mathfrak{A}$.

The most interesting relations of (3.25) are $\{D_+, D_+\} = \beta H_{\mathrm{Sch}}$ and $\{\mathcal{P}^1, \mathcal{P}^1\} = 2I$. We would hope to capture these relations in the Lie colour algebra. Two options for a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra which realise $\{D_+, D_+\}$ or $\{\mathcal{P}^1, \mathcal{P}^1\}$ are $\mathfrak{D}^+$ and $\mathfrak{D}^1$ (respectively).

It is not possible to realise both $\{D_+, D_+\}$ and $\{\mathcal{P}^1, \mathcal{P}^1\}$ simultaneously with a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on at most five basis elements. We can, however, do this with the $\mathbb{Z}_2^3$-graded algebra $\mathfrak{D}$.

3.4.5 Solving the free Lévy–Leblond equation

The $\mathbb{Z}_2^3$-graded algebra $\mathfrak{D}$ contains two operators, $\widehat{P}$ and D_+, which both square to the Schrödinger Hamiltonian H_{Sch}. Finding the eigenstates of either operator would then allow us to solve the Schrödinger equation. However, to solve the Lévy-Leblond equation, we need to find *simultaneous* eigenstates of these operators. In a sense, this couples together these “square roots” of H_{Sch} and leads to the coupling of the components in the solutions to the Lévy-Leblond equation.

Theorem 3.4.2. *Solving the time-independent Lévy-Leblond equation (3.15) with $E > 0$ is equivalent to finding the simultaneous eigenstates for $\widehat{P}$ and D_+ with positive eigenvalues for both operators. Specifically, $|\psi\rangle$ is a solution to the time independent Lévy-Leblond equation with $E > 0$ if and only if*

$$|\psi\rangle = a\,|\varphi_1\rangle + b\mathcal{P}^1\,|\varphi_2\rangle$$

for some simultaneous eigenstates $|\varphi_1\rangle\,, |\varphi_2\rangle$ of $\widehat{P}$ and D_+ with all eigenvalues positive and some constants $a, b \in \mathbb{C}$.

Proof. For the forward direction, let $|\psi\rangle$ be a solution to the time-independent Lévy-Leblond equation. Let $|\chi\rangle = (1/\sqrt{E\beta}\,)\widehat{P}\,|\psi\rangle$.

First, assume $|\chi\rangle = c\,|\psi\rangle$ for some $c \in \mathbb{C}$ then $|\psi\rangle$ is an eigenstate for $\widehat{P}$. From (3.16), we know that

$$\widehat{P}\,|\psi\rangle = (\gamma_+ E + \gamma_- \beta)\,|\psi\rangle = (\gamma_+(\partial_1)^2 + \gamma_-\beta)\,|\psi\rangle = D_+\,|\psi\rangle\,.$$

Therefore, $|\psi\rangle$ is also an eigenstate for D_+. From (3.17) we know that $(\widehat{P})^2\,|\psi\rangle = E\beta\,|\psi\rangle$, and so $c^2 = 1$. If $c = 1$ choose $|\varphi_1\rangle = |\varphi_2\rangle = |\psi\rangle$ and $a = 1$, $b = 0$. If $c = -1$ choose $|\varphi_1\rangle = |\varphi_2\rangle = \mathcal{P}^1\,|\psi\rangle$ and $a = 0$, $b = 1$.

Now, assume $|\chi\rangle \neq c\,|\psi\rangle$ for any $c \in \mathbb{C}$. Set $|\varphi_1\rangle = |\psi\rangle + |\chi\rangle$ and $|\varphi_2\rangle = \mathcal{P}^1(|\psi\rangle - |\chi\rangle)$. From (3.17) we know that $(\widehat{P})^2\,|\psi\rangle = E\beta\,|\psi\rangle$, and so $\widehat{P}\,|\chi\rangle = \sqrt{E\beta}\,\,|\psi\rangle$. It is then easily verified that

$$\widehat{P}\,|\varphi_1\rangle = \sqrt{E\beta}\,\,|\varphi_1\rangle \qquad \text{and} \qquad \widehat{P}\,|\varphi_2\rangle = \sqrt{E\beta}\,\,|\varphi_2\rangle\,.$$

From (3.16), we know that

$$\widehat{P}\,|\psi\rangle = (\gamma_+ E + \gamma_- \beta)\,|\psi\rangle = (\gamma_+(\partial_1)^2 + \gamma_-\beta)\,|\psi\rangle = D_+\,|\psi\rangle\,.$$

Therefore, $|\chi\rangle = (1/\sqrt{E\beta}\,)D_+\,|\psi\rangle$ and a similar argument shows that

$$D_+\,|\varphi_1\rangle = \sqrt{E\beta}\,\,|\varphi_1\rangle \qquad \text{and} \qquad D_+\,|\varphi_2\rangle = \sqrt{E\beta}\,\,|\varphi_2\rangle\,.$$

That is, $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are simultaneous eigenstates for $\widehat{P}$ and D_+ with all eigenvalues positive. Moreover, $|\psi\rangle = |\varphi_1\rangle + \mathcal{P}^1\,|\varphi_2\rangle$ so we set $a = b = 1$.

For the reverse direction, let $|\varphi_1\rangle$ and $|\varphi_2\rangle$ be simultaneous eigenstates of $\widehat{P}$ and D_+ with only positive eigenvalues. Since both operators square to βH_{Sch} we have that $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are also eigenstates of H_{Sch}. Let $E > 0$ be the corresponding eigenvalue of H_{Sch}. Then the eigenvalues for $\widehat{P}$ and D_+ must be $\sqrt{E\beta}$ (note that we are assuming only positive eigenvalues). In particular,

$$-i\partial_1\,|\varphi_1\rangle = \sqrt{E\beta}\,\,|\varphi_1\rangle = \left(-\gamma_+\frac{1}{\beta}(\partial_1)^2 + \gamma_-\beta\right)|\varphi_1\rangle = (\gamma_+ E + \gamma_-\beta)\,|\varphi_1\rangle\,.$$

Rearranging, we get

$$(\gamma_-\beta - \gamma^1 i\partial_1)\,|\varphi_1\rangle = \gamma_+ E\,|\varphi_1\rangle$$

so $|\varphi_1\rangle$ is a solution to the time-independent Lévy-Leblond equation. Note that the eigenvalue of $\mathcal{P}^1|\varphi_2\rangle$ with $\widehat{P}$ and D_+ is $-\sqrt{E\beta}$, so a similar argument shows that $\mathcal{P}^1|\varphi_2\rangle$ is a solution to the Lévy-Leblond equation. Since all the operators are linear, any linear combination of $|\varphi_1\rangle$ and $\mathcal{P}^1|\varphi_2\rangle$ will also be a solution to the Lévy-Leblond equation. □

Theorem 3.4.2 further demonstrates that the operators of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra $\mathfrak{D}$ are fundamental to the free Lévy-Leblond equation.

To conclude this section, we will use Theorem 3.4.2 to solve the Lévy–Leblond equation. So far, all of our results have been representation independent. However, to find a solution we will choose the following matrix representation of the gamma matrices:

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \gamma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The Hilbert space of states is then

$$L^2(\mathbb{R}) \otimes \mathbb{C}^2 = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \mid f, g \in L^2(\mathbb{R}) \right\}.$$

We are looking for simultaneous eigenstates of $\widehat{P} = -i\frac{\partial}{\partial x_1}$ and $D_+ = -\gamma_+(1/\beta)\frac{\partial^2}{\partial x_1^2} + \gamma_-\beta$ with only positive eigenvalues. We will pre-emptively look for eigenstates with eigenvalue $\sqrt{E\beta}$. We know that, up to scaling, the only eigenfunction of the derivative operator with eigenvalue $i\sqrt{E\beta}$ is the exponential function $\exp(i\sqrt{E\beta})$. Therefore, the eigenstates for $\widehat{P}$ are of the form

$$\begin{pmatrix} C_1 e^{i\sqrt{E\beta}\, x_1} \\ C_2 e^{i\sqrt{E\beta}\, x_1} \end{pmatrix}$$

for some constants C_1, $C_2 \in \mathbb{C}$. Now, we examine which of these eigenstates are also eigenstates of D_+:

$$D_+ \begin{pmatrix} C_1 e^{i\sqrt{E\beta}\, x_1} \\ C_2 e^{i\sqrt{E\beta}\, x_1} \end{pmatrix} = \begin{pmatrix} C_2 E e^{i\sqrt{E\beta}\, x_1} \\ C_1 \beta e^{i\sqrt{E\beta}\, x_1} \end{pmatrix} = \frac{C_1\beta}{C_2} \begin{pmatrix} \frac{(C_2)^2 E}{(C_1)^2\beta} C_1 e^{i\sqrt{E\beta}\, x_1} \\ C_2 e^{i\sqrt{E\beta}\, x_1} \end{pmatrix}$$

so we must have $(C_1)^2\beta = (C_2)^2 E$. Therefore, up to scaling, the only simultaneous eigenstate with only positive eigenvalues is

$$\begin{pmatrix} \sqrt{E}\, e^{i\sqrt{E\beta}\, x_1} \\ \sqrt{\beta}\, e^{i\sqrt{E\beta}\, x_1} \end{pmatrix}.$$

By Theorem 3.4.2, the general solution for $E > 0$ is

$$a \begin{pmatrix} \sqrt{E}\, e^{i\sqrt{E\beta}\, x_1} \\ \sqrt{\beta}\, e^{i\sqrt{E\beta}\, x_1} \end{pmatrix} + b\gamma^1\mathcal{P} \begin{pmatrix} \sqrt{E}\, e^{i\sqrt{E\beta}\, x_1} \\ \sqrt{\beta}\, e^{i\sqrt{E\beta}\, x_1} \end{pmatrix} = \begin{pmatrix} a\sqrt{E}\, e^{i\sqrt{E\beta}\, x_1} + b\sqrt{E}\, e^{-i\sqrt{E\beta}\, x_1} \\ a\sqrt{\beta}\, e^{i\sqrt{E\beta}\, x_1} - b\sqrt{\beta}\, e^{-i\sqrt{E\beta}\, x_2} \end{pmatrix}$$

for any a, $b \in \mathbb{C}$. Note that if we substitute $E = 0$, then the above is still an eigenvector. For every $k \in \mathbb{R}$, set

$$\psi_k(x_1) = \frac{\beta^{1/4}}{\sqrt{2\pi(k^2+\beta)}} \begin{pmatrix} k e^{i\sqrt{\beta}\, k x_1} \\ \sqrt{\beta}\, e^{i\sqrt{\beta}\, k x_1} \end{pmatrix}$$

so that $\psi_k(x_1)$ and $\psi_{-k}(x_2)$ are the linearly independent eigenvectors corresponding to $E = k^2$. Note that these eigenvectors are not square-integrable, so do not live inside the Hilbert space $L^2(\mathbb{R}) \otimes \mathbb{C}^2$. If so desired, we could use a rigged Hilbert space (see e.g. [67, 68] and references therein) to make this precise. Regardless, we find that

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}x_1\, \psi_j^\dagger(x_1)\psi_k(x_1) &= \frac{jk+\beta}{\sqrt{(k^2+\beta)(j^2+\beta)}} \frac{\sqrt{\beta}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x_1\, e^{i\sqrt{\beta}\,(k-j)x_1} \\ &= \frac{jk+\beta}{\sqrt{(k^2+\beta)(j^2+\beta)}} \delta(k-j) \\ &= \delta(k-j)\end{aligned} \tag{3.26}$$

where δ is the Dirac delta function. Above, we used the fact that the Fourier transform of δ is the constant function 1. See e.g. [69, Section 2.3] for an introduction to distributions and their Fourier transforms.

Equation (3.26) tells us that the eigenstates ψ_k are orthonormal. The solutions to the time-dependent Lévy-Leblond equation (3.14) corresponding to these eigenvectors are

$$\Psi_k(t, x_1) = e^{-ik^2 t}\psi_k(x_1) = \frac{\beta^{1/4}}{\sqrt{2\pi(k^2+\beta)}} \begin{pmatrix} k e^{i\sqrt{\beta}\,kx_1 - ik^2 t} \\ \sqrt{\beta}\, e^{i\sqrt{\beta}\,kx_1 - ik^2 t} \end{pmatrix}.$$

We can find a more general solution to the time-dependent Lévy-Leblond equation by taking a continuous linear combination:

$$\Psi(t, x_1) = \int_{-\infty}^{\infty} f(k)\Psi_k(t, x_1)\, \mathrm{d}k$$

for some function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Note that, although the eigenstate solutions $\Psi_k(t, \cdot)$ are not in $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ for any $t \in \mathbb{R}$, we can choose f so that the linear combination $\Psi(t, \cdot)$ will be in $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ for each $t \in \mathbb{R}$.

Chapter 4

Finding irreducible colour representations

Lie colour algebras have recently been shown to have relevance to certain physical systems. For example, in [4] the authors found a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry algebra of the Lévy–Leblond equation, and in Chapter 3 we found a $\mathbb{Z}_2^3$-graded Lie colour algebra. Given this, knowledge of the representation theory of Lie colour algebras may become a useful tool. In fact, in [7] the authors used irreducible representations to explore $\mathbb{Z}_2 \times \mathbb{Z}_2$-supermechanics for $\mathcal{N} = 2$, and concluded their paper with the desire for a classification of the irreducible representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$-supersymmetry algebras.

In light of discolouration, it may seem as though the representation theory of Lie colour algebras would be identical to that of Lie superalgebras. However, discolouration only works for Γ-graded representations. And this can be quite different from the more familiar $\mathbb{Z}_2$-graded representation theory for Lie superalgebras.

Nevertheless, in this chapter we will show that the irreducible Γ-graded representations for Lie colour algebras can be derived from the irreducible $\mathbb{Z}_2$-graded representations for Lie superalgebras. This derivation is much more complicated than discolouration, and allows for interesting differences in the representation theories.

We will also show how these results can be strengthened in the case $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ (and more generally $\Gamma = \mathbb{Z}_2^n$). In particular, we obtain bijections $\mathcal{F}_1$ and $\mathcal{F}_2$ that map between equivalence classes of irreducible representations, as per the following diagram.

$$\begin{array}{ccc}
\mathbb{Z}_2\text{-graded superalgebra irreps} & \xleftrightarrow{\quad \mathcal{F}_1 \quad} & \Gamma\text{-graded superalgebra irreps} \\
 & & \text{discolouration} \uparrow \;\; \downarrow \text{recolouration} \\
\text{ungraded colour algebra irreps} & \xleftrightarrow{\quad \mathcal{F}_2 \quad} & \Gamma\text{-graded colour algebra irreps}
\end{array}$$

We end this chapter with an example of applying this process to a colour version of $\mathfrak{sl}_2$.

4.1 Refining extension of a graded representation

Consider a module graded by some group. If we increase the size of the group, we may no longer be able to find a corresponding graded module. However, if we allow ourselves to increase both the size of the group *and* the size of the module, then it would be reasonable to expect that we *could* find a graded module. With this idea in mind, we introduce the notion of a refining extension.

Let $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$ be a Γ-graded Lie colour algebra with commutation factor ε and let H be a subgroup of Γ. Notice that $\mathfrak{g}$ has a natural Γ/H grading, given by $\mathfrak{g} = \bigoplus_{\Lambda\in\Gamma/H} \mathfrak{g}_\Lambda^{\Gamma/H}$ where $\mathfrak{g}_\Lambda^{\Gamma/H} = \bigoplus_{\alpha\in\Lambda} \mathfrak{g}_\alpha$. Recall that $\mathfrak{g}$ also has a natural $\mathbb{Z}_2$-grading

$$\mathfrak{g}^0 = \bigoplus_{\gamma\in\Gamma_0} \mathfrak{g}_\gamma, \qquad \mathfrak{g}^1 = \bigoplus_{\gamma\in\Gamma_1} \mathfrak{g}_\gamma.$$

where $\Gamma_0 = \{\gamma \in \Gamma \mid \varepsilon(\gamma,\gamma) = 1\}$ and $\Gamma_1 = \Gamma \setminus \Gamma_0$. If we choose $H \leq \Gamma_0$, then Γ/H-grading respects the $\mathbb{Z}_2$-grading, i.e.

$$\mathfrak{g}^0 = \bigoplus_{\Lambda\in\Gamma_0/H} \mathfrak{g}_\Lambda^{\Gamma/H}, \qquad \mathfrak{g}^1 = \bigoplus_{\Lambda\in\Gamma_1/H} \mathfrak{g}_\Lambda^{\Gamma/H}.$$

This is useful if we want to prevent H from interfering with discolouration.

Definition 4.1.1. Let $V = \bigoplus_{\Lambda\in\Gamma/H} V_\Lambda$ be a Γ/H-graded $\mathfrak{g}$-module given by ρ. Construct a vector space $V^H = \bigoplus_{\gamma\in\Gamma} V_\gamma^H$, where V_γ^H is isomorphic (as a vector space) to $V_{\gamma+H}$. Let $\varphi_\gamma \colon V_\gamma^H \to V_{\gamma+H}$ be such a vector space isomorphism. The *refining extension of V by H* is the $\mathfrak{g}$-module V^H given by the representation ρ^H defined by

$$\rho^H(x)v = \varphi_{\alpha+\xi}^{-1}(\rho(x)\varphi_\xi(v)), \qquad \text{for } x \in \mathfrak{g}_\alpha,\ v \in V_\xi^H.$$

Intuitively, given a sector V_Λ of V, we add a copy of V_Λ to V^H for every element of the coset Λ. Since Γ is simply the union of all cosets, this gives V^H a natural Γ grading. We then define the action of ρ^H on each copy of V_Λ to be the same as ρ, being careful to respect the new Γ-grading.

Remark 4.1.2. If $\Gamma = K \times H$ for some groups K, then we could equivalently define the refining extension to be $V \otimes \mathbb{F}[H]$ (where $\mathbb{F}[H]$ is the group algebra of H), with representation given by $\rho^H(x)(v \otimes h) = \rho(x)v \otimes (h + \eta)$ for $x \in \mathfrak{g}_{(\kappa,\eta)}$ and $v \in V \otimes \mathbb{F}[H]$. This is similar to the approach used in [53] to classify the simple $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebras. However, with Definition 4.1.1, we do not need to make any additional assumptions on the structure of the group Γ.

As well as a Γ-grading, V^H has a natural Γ/H-grading given by

$$V^H = \bigoplus_{\Lambda\in\Gamma/H} \left(\bigoplus_{\gamma\in\Lambda} V_\gamma^H \right).$$

In this sense, the Γ-grading of V^H is a refinement of the Γ/H-grading.

Proposition 4.1.3. *The $\mathfrak{g}$-module V is a Γ/H-graded submodule of V^H (equivalently, V^H is an extension of V).*

Proof. Define a map $\Phi\colon V \to V^H$ by

$$\Phi(w) = \sum_{\gamma\in\Lambda} \varphi_\gamma^{-1}(w) \qquad \text{for } w \in V_\Lambda,$$

extending to inhomogeneous elements by linearity. Linearity of Φ for homogeneous elements follows from linearity of φ_γ^{-1}. For injectivity, if $\Phi(w) = 0$ then $\varphi_\gamma^{-1}(w) = 0$ for all $\gamma \in \Lambda$ (because each $\varphi_\gamma^{-1}(w)$ is in a different sector of the Γ-grade), so $w = 0$ by injectivity of φ_γ^{-1}. Thus, $\ker\Phi = 0$ and Φ is injective. The map Φ is homogeneous of degree $0 + H$ by definition. Finally, Φ is an intertwiner: for $w \in V_\Lambda$ and $x \in \mathfrak{g}_\alpha$,

$$\begin{aligned}\Phi(\rho(x)w) &= \sum_{\alpha+\gamma\in\alpha+\Lambda} \varphi_{\alpha+\gamma}^{-1}(\rho(x)w)\\ &= \sum_{\alpha+\gamma\in\alpha+\Lambda} \varphi_{\alpha+\gamma}^{-1}(\rho(x)\varphi_\gamma\varphi_\gamma^{-1}(w))\\ &= \sum_{\gamma\in\Lambda} \rho^H(x)\varphi_\gamma^{-1}(w)\\ &= \rho^H(x)\Phi(w).\end{aligned}$$

□

If V can be given a Γ-grading, then we can strengthen the above result:

Proposition 4.1.4. *If V can be given a Γ-grading that respects the Γ/H-grading, then V is a Γ-graded submodule of V^H.*

Proof. Let $V = \bigoplus_{\gamma\in\Gamma} W_\gamma$ be the Γ-grading, with $V_\Lambda = \bigoplus_{\gamma\in\Lambda} W_\gamma$ for all $\Lambda \in \Gamma/H$ (i.e. the Γ-grading respects the Γ/H grading). Define a map $\Phi\colon V \to V^H$ by $\Phi(w) = \varphi_\xi^{-1}(w)$ for $w \in W_\xi$ and extending to inhomogeneous elements by linearity. Linearity and injectivity of Φ follow immediately from linearity and injectivity of φ_ξ^{-1}. The map Φ is homogeneous of degree 0 by definition. And Φ is an intertwiner: for $w \in W_\xi$ and $x \in \mathfrak{g}_\alpha$,

$$\Phi(\rho(x)w) = \varphi_{\alpha+\xi}^{-1}(\rho(x)w) = \varphi_{\alpha+\xi}^{-1}(\rho(x)\varphi_\xi\varphi_\xi^{-1}(w)) = \rho^H(x)\Phi(w).$$

□

The following result is a restatement of the First Isomorphism Theorem in a form that is useful for refining extensions.

Proposition 4.1.5. *Let $V = \bigoplus_{\Lambda\in\Gamma/H} V_\Lambda$ be a Γ/H-graded $\mathfrak{g}$-module given by ρ and $W = \bigoplus_{\gamma\in\Gamma} W_\gamma$ be a Γ-graded $\mathfrak{g}$-module given by π. Suppose there are linear maps $f_\gamma\colon V_{\gamma+H} \to W_\gamma$ that satisfy $f_{\alpha+\gamma}(\rho(x)v) = \pi(x)f_\gamma(v)$ for all $x \in \mathfrak{g}_\alpha$, $v \in V_{\gamma+H}$. Let V^H be the refining extension of V by H and define $f_\gamma^H = f_\gamma\varphi_\gamma$ where $\varphi_\gamma\colon V_\gamma^H \to V_{\gamma+H}$ are the isomorphisms in the construction of V^H. Then,* *(i)* $K = \bigoplus_{\gamma\in\Gamma} \ker f^H_\gamma$ *is a submodule of* V^H*;*

(ii) $I = \bigoplus_{\gamma\in\Gamma} \operatorname{im} f^H_\gamma$ *is a submodule of* W*; and*

(iii) $V^H/K \cong I$.

Proof. Define $f\colon V^H \to W$ by

$$f(v) = f^H_\xi(v)$$

for $v \in V^H_\xi$ and extending to inhomogeneous elements by linearity. It is easy to see that f is an intertwiner, $\ker f = K$ and $\operatorname{im} f = I$. The result follows immediately by applying the First Isomorphism Theorem to f. □

Remark 4.1.6. Refining extensions appear naturally in applications of Lie colour algebras. For instance, the example $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supersymmetric quantum mechanical system presented in [29, Section III] can be obtained by applying a cocycle twist to the refining extension of Witten's model [30] (with an additional central charge) by $\mathbb{Z}_2$.

4.2 Finding irreducible graded representations

Let $\mathfrak{g}$ be a Lie colour algebra. We will show that all the irreducible Γ-graded representations can be derived from the irreducible Γ/H-graded representations. If we choose $H = \Gamma_0$ so that $\Gamma/H \le \mathbb{Z}_2$ (and discolour/recolour the algebra as needed) then this will allow us to apply results from the well-studied $\mathbb{Z}_2$-graded representation theory of Lie algebras and superalgebras to the representation theory of colour algebras.

Suppose we are given a finite-dimensional irreducible Γ/H-graded $\mathfrak{g}$-module V. To construct an irreducible Γ-graded $\mathfrak{g}$-module, we can simply take the refining extension V^H and quotient out by a maximal Γ-graded submodule. Since V^H may have many maximal submodules, this procedure may yield many nonisomorphic Γ-graded modules for each Γ/H-graded module. However, in the following theorem, we show that every irreducible Γ-graded $\mathfrak{g}$-module can be constructed in this way.

Theorem 4.2.1. *Let* $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$ *be a Lie colour algebra. Let* $W = \bigoplus_{\gamma\in\Gamma} W_\gamma$ *be a finite-dimensional irreducible* Γ*-graded* $\mathfrak{g}$*-module. Then, for* $H \le \Gamma$*, there exists an irreducible* Γ/H*-graded* $\mathfrak{g}$*-module* V *such that* W *is a graded quotient of the refining extension of* V *by* H*.*

Proof of Theorem 4.2.1. Note that W has a natural Γ/H-grading given by

$$\bigoplus_{\Lambda\in\Gamma/H} \left(\bigoplus_{\gamma\in\Lambda} W_\gamma \right).$$

Let V be an irreducible Γ/H-graded submodule of W. Such a V exists by strong induction on the dimension of W. If W is one-dimensional then we can choose $V = W$. If W has dimension larger than one, then either W is irreducible as a Γ/H-graded $\mathfrak{g}$-module (in which case, we choose $V = W$), or W has a Γ/H-graded submodule of strictly smaller dimension to which the inductive hypothesis applies.

Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(W, \varepsilon)$ be the representation corresponding to W (ρ also provides a submodule structure for V).

Let $f_\xi\colon V_{\xi+H} \to W_\xi$ be the projection of $V_{\xi+H}$ onto W_ξ, so that $v = \sum_{\gamma\in\Lambda} f_\gamma(v)$ for all $v \in V_\Lambda$, $\Lambda \in \Gamma/H$. Note that f_ξ is well-defined because of the direct sum decomposition $V_{\xi+H} \subseteq \bigoplus_{\gamma\in\xi+H} W_\gamma$. In particular, every $v \in V_{\xi+H}$ can be written as a linear combination $\sum_{\gamma\in\xi+H} w_\gamma$ for unique $w_\gamma \in W_\gamma$, so the unique choice for $f_\xi(v)$ is w_ξ. Additionally, it is easy to see that f_ξ is linear.

Let $v \in V_\Lambda$ and $x \in \mathfrak{g}_\alpha$. Since $v = \sum_{\gamma\in\Lambda} f_\gamma(v)$, we have $\rho(x)v = \sum_{\gamma\in\Lambda} \rho(x) f_\gamma(v)$ (note that $\rho(x)f_\gamma(v) \in W_{\alpha+\gamma}$ for all $\gamma \in \Gamma$). Since such a linear combination of homogeneous elements is unique, for all $\gamma \in \Lambda$ we have that $f_{\alpha+\gamma}(\rho(x)v) = \rho(x)f_\gamma(v)$ by the definition of $f_{\alpha+\gamma}$.

Therefore, by Proposition 4.1.5, $I \cong V^H/K$ for $I = \bigoplus_{\gamma\in\Gamma} \operatorname{im} f_\gamma^H$ and $K = \bigoplus_{\gamma\in\Gamma} \ker f_\gamma^H$. But I is a Γ-graded is a submodule of W, so $I = 0$ or $I = W$ by irreducibility of W. However, since $V \neq 0$, there is some nonzero sector V_Λ of V (with $\Lambda \in \Gamma/H$). Any non-zero element $v \in V_\Lambda$ can be written as $v = \sum_{\gamma\in\Lambda} f_\gamma(v)$, so $f_\xi(v) \neq 0$ for at least one $\xi \in \Gamma$ (otherwise $v = 0$). That is, $\operatorname{im} f_\xi \neq 0$, so $I \neq 0$. We conclude that $I = W$; hence $W \cong V^H/K$ as claimed. □

Theorem 4.2.1 provides us with a general method of obtaining the irreducible Γ-graded modules from the Γ/H-graded ones. However, this theorem does not provide much information about the actual structure of the Γ-graded modules.

4.3 The case $H \cong \mathbb{Z}_2$

If we limit our discussion to the case where $H \cong \mathbb{Z}_2$, then the small size of the group drastically limits the possibilities, and we can deduce much stronger results about the structure of the Γ-graded modules. In particular, we will construct a bijection between (equivalence classes of) the Γ/H- and Γ-graded modules and show that every irreducible Γ-graded module is either an irreducible Γ/H-graded module or its refining extension.

To actually construct such a bijection, we need to know the composition series for V^H and the amount of freedom we have when choosing a Γ-grading for V (when such a grading is possible). With these goals in mind, we introduce the notions of Γ/H-representative negation and change of H-parity.

4.3.1 Γ/H-representative negation

The composition factors for V^H (as a Γ/H-graded module) restrict which quotients are possible, and hence which Γ-graded representations are possible. Given that V^H was constructed from two ($= |H|$) copies of V, we might expect that the composition factors are merely V, V. This is not the case; the composition factors are V^-, V for some module V^- which we call the

Γ/H-representative negation. The structure of V^- is almost identical to that of V, with the representation differing only by an occasional sign change.

More precisely, let G be a set containing exactly one representative from each coset in Γ/H. Note that G does not necessarily form a group. Let $V = \bigoplus_{\Lambda\in\Gamma/H} V_\Lambda$ be a Γ/H-graded $\mathfrak{g}$-module given by the representation ρ. Define a new module V^- with representation ρ^- via the following: Set $V^- = \bigoplus_{\Lambda\in\Gamma/H} V_\Lambda^-$ where $V_\Lambda^- = V_\Lambda$ (as vector spaces). Take an arbitrary $x \in \mathfrak{g}_\alpha$, $\alpha \in \Gamma$ and $v \in V_\Lambda$, $\Lambda \in \Gamma/H$. Let ξ be the unique representative of Λ that is also in G. Define ρ^- by

$$\rho^-(x)v = \begin{cases} \rho(x)v & \text{if } \alpha+\xi \in G \\ -\rho(x)v & \text{if } \alpha+\xi \notin G. \end{cases}$$

Proposition 4.3.1. *V^- is a $\mathfrak{g}$-module.*

Proof. Let $H = \{0, \eta\}$. Both linearity properties of ρ^- follow immediately from that of ρ.

Let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$ and $v \in V_\Lambda$ be arbitrary. Let ξ be the unique representative of Λ that is also in G. We claim that $\rho^-(x)\rho^-(y)v = \rho(x)\rho(y)v$ if $\alpha+\beta+\xi \in G$ and $\rho^-(x)\rho^-(y)v = -\rho(x)\rho(y)v$ if $\alpha+\beta+\xi \notin G$. If $\beta+\xi \in G$ then $\rho^-(y)v = \rho(y)$ and the claim is obvious. If $\beta+\xi \notin G$ then $\rho^-(y)v = -\rho(y)$. Additionally, the representative of $\beta+\xi+H$ in G must be $\beta+\xi+\eta$. Thus, by definition,

$$\rho^-(x)\rho^-(y)v = -\rho^-(x)(\rho(y))v = \begin{cases} -\rho^-(x)\rho^-(y) & \text{if } \alpha+\beta+\xi+\eta \in G \\ \rho^-(x)\rho^-(y) & \text{if } \alpha+\beta+\xi+\eta \notin G \end{cases}.$$

Since $\alpha+\beta+\xi \in G$ if and only if $\alpha+\beta+\xi+\eta \notin G$, this proves the claim.

Now, ρ^- is a representation:

$$\begin{aligned} \rho^-([\![x,y]\!])v &= \begin{cases} \rho([\![x,y]\!])v & \text{if } \alpha+\beta+\xi \in G \\ -\rho([\![x,y]\!])v & \text{if } \alpha+\beta+\xi \notin G \end{cases} \\ &= \begin{cases} (\rho(x)\rho(y) - \varepsilon(\alpha,\beta)\rho(y)\rho(x))v & \text{if } \alpha+\beta+\xi \in G \\ -(\rho(x)\rho(y) - \varepsilon(\alpha,\beta)\rho(y)\rho(x))v & \text{if } \alpha+\beta+\xi \notin G \end{cases} \\ &= (\rho^-(x)\rho^-(y) - \rho^-(y)\rho^-(x))v \end{aligned}$$

by the above claim. □

We call V^- the *Γ/H-representative negation of V (with representatives G)*. The module V^- is not necessarily isomorphic to V. Note that the Γ/H-representative negation of V^- is V.

Proposition 4.3.2. *If V is irreducible then V^- is irreducible.*

Proof. If U is a submodule of V^- then $\rho^-(\mathfrak{g})U \subseteq U$. Since ρ only differs from ρ^- by a sign (at most), we have that $\rho(\mathfrak{g})U \subseteq U$. Hence, U is also submodule of V. If V is irreducible then, as a vector space, $U = 0$ or $U = V = V^-$. □

Proposition 4.3.3. *There is an equality of $\mathfrak{g}$-modules: $(V^H)^- = (V^-)^H$.*

Proof. Clearly $(V^H)^-$ and $(V^-)^H$ are equal as vector spaces. We just need to check the module structure. Let ρ^{H-} be the representation for $(V^H)^-$ and ρ^{-H} be the representation for $(V^-)^H$. Let $v \in V_\gamma^H$ and $x \in \mathfrak{g}_\alpha$. Choose ξ to be the unique representative of $\gamma + H$ that is also in G. Then,

$$\begin{aligned}
\rho^{H-}(x)v &= \begin{cases} \rho^H(x)v & \text{if } \alpha+\xi \in G \\ -\rho^H(x)v & \text{if } \alpha+\xi \notin G \end{cases} \\
&= \begin{cases} \varphi_{\alpha+\gamma}^{-1}\rho(x)\varphi_\gamma v & \text{if } \alpha+\xi \in G \\ \varphi_{\alpha+\gamma}^{-1}(-\rho(x))\varphi_\gamma v & \text{if } \alpha+\xi \notin G \end{cases} \\
&= \varphi_{\alpha+\gamma}^{-1}\rho^-(x)\varphi_\gamma v \\
&= \rho^{-H}(x)v.
\end{aligned}$$

□

It is clear from the definition and the above propositions that the Γ/H-representative negation has a very similar module structure to V. This is useful when examining the structure of V^H:

Proposition 4.3.4. *$V^- \cong V^H/V$.*

Proof. Let $H = \{0, \eta\}$. Recall that V is a Γ/H-graded submodule of V^H by Proposition 4.1.3, with embedding $\Phi\colon V \to V^H$given by

$$\Phi(w) = \sum_{\gamma\in\Lambda} \varphi_\gamma^{-1}(w)$$

for $w \in V_\Lambda$ (notation as in Section 4.1). Define a map $\Psi\colon V^- \to V^H/\Phi(V)$ given by

$$\Psi(w) = \varphi_\xi^{-1}(w) + \Phi(V_\Lambda)$$

for $w \in V_\Lambda^-$ and ξ the unique representative of Λ in the set G. Define Ψ for inhomogeneous elements by linearity. We claim that Ψ is an isomorphism. Indeed, linearity for homogeneous elements follows from the fact that $\varphi_\gamma\colon V_\gamma^H \to V_{\gamma+H}$ is a vector space isomorphism for all $\gamma \in \Gamma$.

For injectivity, if $\Psi(w) = 0 = 0 + \Phi(V_\Lambda)$ then $\Psi(w) \in \Phi(V_\Lambda)$; that is,

$$\varphi_\xi^{-1}(w) = \varphi_\xi^{-1}(w) + \varphi_{\xi+\eta}^{-1}(w) \qquad \Longrightarrow \qquad 0 = \varphi_{\xi+\eta}^{-1}(w)$$

(noting that $\Lambda = \{\xi, \xi+\eta\}$) and hence $w = 0$ by injectivity of $\varphi_{\xi+\eta}^{-1}$. Thus, $\ker\Psi = 0$ and Ψ is injective.

For surjectivity, let $v + \Phi(V_\Lambda)$ be an arbitrary element of $(V^H/\Phi(V))_\Lambda = (V_\xi^H \oplus V_{\xi+\eta}^H)/\Phi(V_\Lambda)$. Write $v = v_\xi + v_{\xi+\eta}$ where $v_\xi \in V_\xi^H$, $v_{\xi+\eta} \in V_{\xi+\eta}^H$. Then,

$$\begin{aligned}
\Psi(\varphi_\xi(v_\xi) - \varphi_{\xi+\eta}(v_{\xi+\eta})) &= (v_\xi - \varphi_\xi\varphi_{\xi+\eta}(v_{\xi+\eta})) + \Phi(V_\Lambda) \\
&= (v_\xi - \varphi_\xi^{-1}\varphi_{\xi+\eta}(v_{\xi+\eta})) + \Phi(\varphi_{\xi+\eta}(v_{\xi+\eta})) + \Phi(V_\Lambda) \\
&= (v_\xi - \varphi_\xi^{-1}\varphi_{\xi+\eta}(v_{\xi+\eta})) + (\varphi_\xi^{-1}\varphi_{\xi+\eta}(v_{\xi+\eta}) + v_{\xi+\eta}) + \Phi(V_\Lambda) \\
&= v + \Phi(V_\Lambda)
\end{aligned}$$

so Ψ is surjective.

The map Ψ is homogeneous of degree $0 + \Phi(V)$ by definition. Finally, Ψ is an intertwiner: for $w \in V_\Lambda^-$ and $x \in \mathfrak{g}_\alpha$, if $\alpha + \xi \in G$ then

$$\begin{aligned}\Psi(\rho^-(x)w) &= \varphi_{\alpha+\xi}^{-1}(\rho(x)w) + \Phi(V) \\ &= \varphi_{\alpha+\xi}^{-1}\rho(x)\varphi_\xi\varphi_\xi^{-1}(w) + \Phi(V) \\ &= \rho^H(x)\Psi(w)\end{aligned}$$

and if $\alpha + \xi \notin G$ then $\alpha + \xi + \eta \in G$ and so

$$\begin{aligned}\Psi(\rho^-(x)w) &= \varphi_{\alpha+\xi+\eta}^{-1}(-\rho(x)w) + \Phi(V) \\ &= -\varphi_{\alpha+\xi+\eta}^{-1}(\rho(x)w) + \Phi(\rho(x)w) + \Phi(V) \\ &= -\varphi_{\alpha+\xi+\eta}^{-1}\rho(x)w + (\varphi_{\alpha+\xi}^{-1}\rho(x)w + \varphi_{\alpha+\xi+\eta}^{-1}\rho(x)w) + \Phi(V) \\ &= \varphi_{\alpha+\xi}^{-1}\rho(x)w + \Phi(V) \\ &= \varphi_{\alpha+\xi}^{-1}\rho(x)\varphi_\xi\varphi_\xi^{-1}(w) + \Phi(V) \\ &= \rho^H(x)\Psi(w).\end{aligned}$$

In either case, we have that Ψ is an intertwiner. □

Remark 4.3.5. The above proof relies on the fact that $H \cong \mathbb{Z}_2$.

The following corollary shows that the Γ/H-representative negation of V does not depend on the choice of representatives G.

Corollary 4.3.6. *Let G, $\widetilde{G}$ be two sets which each contain exactly one representative from each coset of Γ/H. Let V^-, $\widetilde{V}^-$ be the corresponding Γ/H-representative negations of V. Then V^- and $\widetilde{V}^-$ are isomorphic as Γ/H-graded modules.*

Proof. $V^- \cong V^H/V \cong \widetilde{V}^-$. □

If we assume that V is irreducible, then we can determine the composition series for V^H in terms of V and V^-.

Corollary 4.3.7. *Assume V is irreducible. As a Γ/H-module, the composition series for the refining extension V^H is*

$$V^H \supseteq V \supseteq 0$$

with composition factors V^-, V.

Proof. We have $V/0 \cong V$ and $V^H/V \cong V^-$. Using Proposition 4.3.2, both V^- and V are irreducible, so the given chain of submodules is indeed a composition series. □

4.3.2 Change of H-parity

It is possible for two modules to be non-isomorphic as Γ-graded modules but isomorphic as Γ/H-graded modules. It is necessary to consider this fact when trying to construct all the Γ-graded modules from the Γ/H-graded ones. A common way (for $H \cong \mathbb{Z}_2$, the only way) that non-isomorphic Γ-gradings appear is by permuting the newly added sectors by elements of H. Such a permutation is called a change of H-parity.

We say that two Γ-graded modules $W = \bigoplus_{\gamma\in\Gamma} W_\gamma$ and $W' = \bigoplus_{\gamma\in\Gamma} W'_\gamma$ are *equivalent up to change of H-parity* if there exists a Γ/H-graded isomorphism $\Phi\colon W \to W'$ such that $\Phi(W_\xi) = W'_{\xi+h}$ for some $h \in H$ and each $\xi \in \Gamma$. It is easily verified that this is an equivalence relation. Also, note that Γ-graded isomorphism is a special case of change of H-parity; i.e. if W and W' are isomorphic as Γ-graded modules then W and W' are equivalent up to change of H-parity of sectors.

Lemma 4.3.8. *If V is a finite-dimensional irreducible Γ/H-graded $\mathfrak{g}$-module, then two Γ-gradings for V are equivalent up to change of H-parity.*

More precisely, this lemma tells us that if $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$ and $V = \bigoplus_{\gamma\in\Gamma} V'_\gamma$ are two Γ-gradings for V which both respect the Γ/H grading (i.e. for each $\Lambda \in \Gamma/H$ the original Λ-sector is $\bigoplus_{\gamma\in\Lambda} V_\gamma = \bigoplus_{\gamma\in\Lambda} V'_\gamma$) then there exists an isomorphism $\Phi\colon V \to V$ such that $\Phi(V_\xi) = V'_{\xi+h}$ for some $h \in H$ and for each $\xi \in \Gamma$.

Proof. Define the map $\Psi\colon V \to V$ to be the projection of V_ξ onto V'_ξ for each $\xi \in \Gamma$, and then extend to inhomogeneous elements. That is, if $v \in V_\xi$ and $v = \sum_{\gamma\in\xi+H} v'_\gamma$ for $v'_\gamma \in V'_\gamma$, then $\Psi(v) = v'_\xi$. Since the above linear combination is unique, Ψ is well-defined. Clearly, Ψ is linear. Since $\Psi(\bigoplus_{\gamma\in\Lambda} V_\gamma) = \bigoplus_{\gamma\in\Lambda} V'_\gamma$, we have that Ψ is homogeneous of degree 0 (using the assumption that the Γ-gradings respect the Γ/H-grading). For $v \in V_\xi$ and $x \in \mathfrak{g}_\alpha$, if $v = \sum_{\gamma\in\xi+H} v'_\gamma$ then

$$\rho(x)v = \sum_{\gamma\in\xi+H} \rho(x)v'_\gamma,$$

and since $\rho(x)v \in V_{\alpha+\xi}$ and $\rho(x)v'_\xi \in V'_{\alpha+\xi}$, we have that $\Psi(\rho(x)v) = \rho(x)v'_\xi = \rho(x)\Psi(v)$. Thus, Ψ is an intertwiner.

Now, $\Psi(V)$ is a submodule of V, so $\Psi(V) = 0$ or $\Psi(V) = V$ since V is irreducible. If $\Psi(V) = V$ then Ψ is surjective and hence injective (since V is finite dimensional). That is, Ψ is an isomorphism. Additionally, $\Psi(V_\xi) = V'_\xi$, so choosing $\Phi = \Psi$ completes the proof in this case. Alternatively, if $\Psi(V) = 0$ then

$$V_\xi \subseteq \bigoplus_{\substack{\gamma\in\xi+H\\ \gamma\neq\xi}} V'_\gamma = V'_{\xi+\eta}$$

where $H = \{0, \eta\}$. In this case, choosing Φ to be the identity map completes the proof. □

Remark 4.3.9. The above proof relies on the fact that $H \cong \mathbb{Z}_2$.

Note that is always possible to change the H-parity of a module. However, changing the H-parity might yield a module isomorphic to the original. Indeed, changing the H-parity of V^H yields V^H, i.e. there is only one element in the equivalence class of V^H (up to isomorphism).

4.3.3 A bijection

Let $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$ be a Lie colour algebra. Additionally, let $H \leq \Gamma$ with $H \cong \mathbb{Z}_2$. Let $\mathsf{Irr}^-_{\Gamma/H}(\mathfrak{g})$ be the collection of equivalence classes of finite-dimensional irreducible Γ/H-graded $\mathfrak{g}$-modules, where two Γ/H-graded modules are equivalent if, after potentially performing Γ/H-representative negation, they are isomorphic. Let $\mathsf{Irr}^{H\text{p}-}_{\Gamma}(\mathfrak{g})$ be the collection of equivalence classes of finite-dimensional irreducible Γ-graded $\mathfrak{g}$-modules, where two Γ-graded modules are equivalent if, after potentially performing Γ/H-representative negation, they are equivalent up to change of H-parity.

Theorem 4.3.10. *Define a function $\mathcal{F}\colon \mathsf{Irr}^-_{\Gamma/H} \to \mathsf{Irr}^{H\text{p}-}_{\Gamma}$ by*

$$\mathcal{F}(V) = V^H/K$$

where V and V^H/K are representatives of the equivalence classes, and K is a maximal submodule of V^H. Then $\mathcal{F}$ is well-defined (does not depend on the choice of representative V nor the choice of K) and bijective.

Proof. From Corollary 4.3.7, we have the following composition series of Γ/H-graded modules:

$$V^H \supseteq V \supseteq 0$$

with composition factors V^-, V (where V^- is the Γ/H-representative negation of V). Applying the Schreier Refinement Theorem to this composition series and $V^H \supseteq K \supseteq 0$, we have that (as a Γ/H-module) K is isomorphic to 0, V or V^-.

If $K = 0$ then this is the unique choice for K by the definition of a maximal submodule. In this case, $\mathcal{F}(V) = V^H$.

Now, suppose $K \cong V$ as Γ/H-graded modules. Then, 0 cannot be a maximal submodule, so all maximal submodules must be isomorphic to V or V^-. For now, assume that $K \cong V$. Note that K gives V a Γ-grading. In particular, we choose a Γ-grading $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$ such that there exists an isomorphism of Γ/H-graded modules $\Phi\colon K \to V$ with $\Phi(K_\gamma) = V_{\gamma+\eta}$ for nonzero $\eta \in H$ and all $\gamma \in \Gamma$ (where $K = \bigoplus_{\gamma\in\Gamma} K_\gamma$). Given a Γ-grading, there is a natural embedding of V into V^H (cf. Proposition 4.1.4). Since $V^H_\gamma \cong V_{\gamma+H} \cong V_\gamma \oplus V_{\gamma+\eta} \cong V_\gamma \oplus K_\gamma$ (vector space isomorphisms), we find that V^H/K is isomorphic as a Γ-graded module to $V^- = \bigoplus_{\gamma\in\Gamma} V^-_\gamma$. If we instead assume that $K \cong V^-$, a similar argument shows that V^H/K is isomorphic as a Γ-graded module to V. Since V is equivalent to V, and all Γ-gradings of V or V^- are equivalent up to change of H-parity by Lemma 4.3.8, we find that $\mathcal{F}(V) = \bigoplus_{\gamma\in\Gamma} V^-_\gamma$ does not depend on the choice of K.

Obviously $\mathcal{F}$ does not depend on the choice of representative V. Thus, $\mathcal{F}$ is well-defined. That $\mathcal{F}$ is surjective follows immediately from Theorem 4.2.1.

For injectivity, assume $\mathcal{F}(V) = \mathcal{F}(V')$. From the above discussion, we can see that $\mathcal{F}(V) = V^H$, $\mathcal{F}(V) = V$ or $\mathcal{F}(V) = V^-$ and similarly for $\mathcal{F}(V')$. We analyse the possibilities case-by-case. We cannot have $\mathcal{F}(V) = V^H$ and $\mathcal{F}(V') = V'$ because, as Γ/H-graded modules, the former is reducible but the latter is irreducible. Similarly, we cannot have $\mathcal{F}(V) = V^H$ and $\mathcal{F}(V') = V'^-$; nor $\mathcal{F}(V) = V$ and $\mathcal{F}(V') = (V')^H$; nor $\mathcal{F}(V) = V^-$ and $\mathcal{F}(V') = (V')^H$. If $\mathcal{F}(V) = V$ and $\mathcal{F}(V') = V'$ then, by the definition of equivalence up to change of H-parity, V and V' must be isomorphic as Γ/H-graded modules (as required). Similarly, V and V' are equivalent if $\mathcal{F}(V) = V^-$ and $\mathcal{F}(V) = V'^-$; or $\mathcal{F}(V) = V$ and $\mathcal{F}(V') = V'^-$; or $\mathcal{F}(V) = V^-$ and $\mathcal{F}(V') = V'$. If $\mathcal{F}(V) = V^H$ and $\mathcal{F}(V') = (V')^H$ then V^H is isomorphic as a Γ/H-graded module to $(V')^H$ or $((V')^H)^- = (V'^-)^H$ (using Proposition 4.3.3). The composition factors of V^H are V^-, V. And $(V')^H$ and $(V'^-)^H$ both have the same composition factors: V'^-, V'. Therefore, by the Jordan–Hölder Theorem, we must have that V is isomorphic as a Γ/H-graded module to V'^- or V'. That is V is equivalent to V'. □

Corollary 4.3.11. *Let $H \cong \mathbb{Z}_2$. Then, every finite-dimensional irreducible Γ-graded module is either irreducible as a Γ/H-graded irreducible module or is the refining extension of an irreducible Γ/H-graded module.*

Proof. We examine the possible cases encountered in the proof of the above theorem. □

Taking $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$, $H = \{00, 11\}$, the above theorem (in combination with discolouration/recolouration) gives a way to construct the irreducible modules of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie superalgebras from the $\mathbb{Z}_2$-graded Lie superalgebras. In particular, Theorem 4.3.10 gives us a bijection between $\mathbb{Z}_2$-graded superalgebras representations and Γ-graded superalgebra representations which we can then recolour. Using iterated applications of the theorem gives a procedure to produce irreducible $\mathbb{Z}_2^n$-graded modules for any positive integer n. For $\Gamma = \mathbb{Z}_2^n$, Theorem 4.3.10 also gives us a bijection between the graded and ungraded modules for the colour algebra.

4.4 Example: colour $\mathfrak{sl}_2$

As an example, we will determine the finite-dimensional irreducible representations of $\mathfrak{sl}_2^c$ over $\mathbb{C}$. The algebra $\mathfrak{sl}_2^c$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded colour Lie algebra with commutation factor $\varepsilon(\alpha_1\alpha_2, \beta_1\beta_2) = (-1)^{\alpha_1\beta_2 - \alpha_2\beta_1}$ and sectors spanned by the following elements:

00-sector: 0 10-sector: a_1 01-sector: a_2 11-sector: a_3

and equipped with a bracket $[\![\cdot, \cdot]\!]$ which satisfies

$$[\![a_1, a_2]\!] = a_3 \qquad [\![a_2, a_3]\!] = a_1 \qquad [\![a_3, a_1]\!] = a_2.$$

Using the multiplier $\sigma(\alpha_1\alpha_2, \beta_1\beta_2) = (-1)^{\alpha_2\beta_1}$, $\mathfrak{sl}_2^c$ discolours to a Lie algebra with Lie bracket $[\![\cdot,\cdot]\!]^\sigma = [\cdot,\cdot]$ given by

$$[a_1, a_2] = a_3 \qquad [a_2, a_3] = -a_1 \qquad [a_3, a_1] = -a_2.$$

These commutation relations are satisfied by the following realisation

$$a_1 = \frac{i}{2}(e - f) \qquad a_2 = -\frac{1}{2}(e + f) \qquad a_3 = -\frac{i}{2}h. \tag{4.1}$$

in terms of a standard basis $\{h, e, f\}$ for $\mathfrak{sl}_2$, ($[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$). Consequently, the discolouration $(\mathfrak{sl}_2^c)^\sigma$ is isomorphic to $\mathfrak{sl}_2$ (hence the notation).

The Lie colour algebra $\mathfrak{sl}_2^c$ has appeared in the literature before. In [59], the authors found all the ungraded irreducible representations for $\mathfrak{sl}_2^c$ by embedding $U(\mathfrak{sl}_2^c)$ in $M_2(U(\mathfrak{sl}_2))$ (the space of 2×2 matrices with entries in $U(\mathfrak{sl}_2)$). The approach that we will take via refining extensions is very similar, though we will not work at the level of universal enveloping algebras. Additionally, our approach is undertaken in the context of the more general preceding sections.

Our strategy is as follows:

1. find all of the ungraded irreducible representations for $\mathfrak{sl}_2$;
2. use the techniques from the preceding sections to find the $\mathbb{Z}_2$-graded irreducible representations for $\mathfrak{sl}_2$;
3. similarly, find the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded irreducible representations for $\mathfrak{sl}_2$;
4. recolour to obtain the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded irreducible representations for $\mathfrak{sl}_2^c$.
5. Find the ungraded irreducible representations for $\mathfrak{sl}_2^c$ as subrepresentations

4.4.1 Irreducible representations for $\mathfrak{sl}_2$

Recall that $\mathfrak{sl}_2$ has a unique irreducible representation V_λ of dimension $\lambda + 1$ for every highest weight $\lambda \in \mathbb{Z}_{\geq 0}$ (see e.g. [70, Lemma 3.2]). We will choose a basis $\{v_0, v_1, \ldots, v_\lambda\}$ for V_λ such that

$$hv_j = (\lambda - 2j)v_j \qquad ev_j = (\lambda - j + 1)v_{j-1} \qquad fv_j = (j + 1)v_{j+1}$$

with the convention $v_{-1} = v_{\lambda+1} = 0$. In particular, note that v_0 is the highest weight vector. (In the above line and what follows, we have omitted the explicit representation map $\rho\colon \mathfrak{sl}_2 \to \mathfrak{gl}(V_\lambda)$.) Using (4.1), we find that

$$\begin{aligned}
a_1 v_j &= \frac{i}{2}((\lambda - j + 1)v_{j-1} - (j + 1)v_{j+1}) \\
a_2 v_j &= -\frac{1}{2}((\lambda - j + 1)v_{j-1} + (j + 1)v_{j+1}) \\
a_3 v_j &= -\frac{i}{2}(\lambda - 2j)v_j.
\end{aligned}$$

4.4.2 $\mathbb{Z}_2$-graded irreducible representations

We wish to find the $\mathbb{Z}_2$-graded representations. So, we choose $\Gamma_1 = H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2/\{00, 11\} \cong \mathbb{Z}_2$. Under this group and the original $\mathfrak{sl}_2^c$ grading, the sectors for $\mathfrak{sl}_2$ are spanned by the following elements:

0-sector: a_3 1-sector: a_1, a_2.

The ungraded module V_λ can be given a Γ_1-grading: $V^{\mathrm{E}}_{\lambda,0} \oplus V^{\mathrm{E}}_{\lambda,1}$ where

$$V^{\mathrm{E}}_{\lambda,0} = \mathrm{span}\{v_j \mid j \text{ even}\} \qquad V^{\mathrm{E}}_{\lambda,1} = \mathrm{span}\{v_j \mid j \text{ odd}\}.$$

We can easily check that $a_1 V^{\mathrm{E}}_{\lambda,\gamma} \subseteq V^{\mathrm{E}}_{\lambda,\gamma+1}$, $a_2 V^{\mathrm{E}}_{\lambda,\gamma} \subseteq V^{\mathrm{E}}_{\lambda,\gamma+1}$ and $a_3 V^{\mathrm{E}}_{\lambda,\gamma} \subseteq V^{\mathrm{E}}_{\lambda,\gamma}$ for $\gamma \in \mathbb{Z}_2$. Call this Γ_1-graded module V^{E}_λ.

Since V_λ is an irreducible ungraded module, V^E_λ is an irreducible Γ_1-graded module. By changing the H_1-parity of V^{E}_λ, we get another Γ_1-graded irreducible module $V^{\mathrm{O}}_\lambda = V^{\mathrm{O}}_{\lambda,0} \oplus V^{\mathrm{O}}_{\lambda,1}$ where

$$V^{\mathrm{O}}_{\lambda,0} = \mathrm{span}\{v_j \mid j \text{ odd}\} \qquad V^{\mathrm{O}}_{\lambda,1} = \mathrm{span}\{v_j \mid j \text{ even}\}.$$

By Lemma 4.3.8, we do not need to search for any other gradings. And by Corollary 4.3.11, we do not need to check the refining extension at all. Thus, the finite-dimensional irreducible Γ_1-graded $\mathfrak{sl}_2$-modules are V^{E}_λ and V^{O}_λ for $\lambda \in \mathbb{Z}_{\geq 0}$.

4.4.3 $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded irreducible representations

Now we wish to find the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded representations. So, we choose $\Gamma_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $H_2 = \{00, 11\}$. To try and find Γ_2-gradings, it is first useful to know what properties they must satisfy.

Lemma 4.4.1. *Let $V_\lambda = \bigoplus_{\gamma\in\Gamma_2} W_\gamma$ be a Γ_2-grading for V_λ. Then $v_0 = \sum_{\gamma\in\Gamma_2} v_{0,\gamma}$ where $v_{0,\gamma} \in W_\gamma$ is some linear combination of v_0 and v_λ (possibly equal to zero).*

Proof. Since $h = 2ia_3$ is homogeneous of degree 11, we find that $h^2 W_\gamma \subseteq W_\gamma$. If we project v_0 onto each sector and write $v_0 = \sum_{\gamma\in\Gamma_2} v_{0,\gamma}$, for $v_{0,\gamma} \in W_\gamma$ then we find that

$$\sum_{\gamma\in\Gamma_2} h^2 v_{0,\gamma} = h^2(v_0) = \sum_{\gamma\in\Gamma_2} \lambda^2 v_{0,\gamma}.$$

In particular, $h^2 v_{0,\gamma} = \lambda^2 v_{0,\gamma}$ for every $\gamma \in \Gamma_2$. Using our knowledge of the eigenvectors of h, we find that $v_{0,\gamma}$ must be a linear combination of v_0 and v_λ. □

Remark 4.4.2. A similar result holds for $v_\lambda = \sum_{\gamma\in\Gamma_2} v_{\lambda,\gamma}$.

Lemma 4.4.1 tells us that at least one sector contains a nonzero linear combination of v_0 and v_λ. Our strategy is to then apply a_1 and a_2 to this linear combination to help deduce the structure of a Γ_2-grading or prove that one cannot exist.

The λ even case

If λ is even, then both v_0 and v_λ are in $V^{\mathrm{E}}_{\lambda,0}$. Therefore, using Lemma 4.4.1, we find that the 00-sector or 11-sector must contain a nonzero linear combination of v_0 and v_λ. We choose (after some trial and error) $v_0 + v_\lambda$ to be in the 00-sector and $v_0 - v_\lambda$ to be in the 11-sector. With this choice, we can find a Γ_2-grading for V^{E}_λ:

$$\begin{aligned} V^{\mathrm{E}+}_{\lambda,00} &= \operatorname{span}\{v_j + v_{\lambda-j} \mid j \text{ even}\}, & V^{\mathrm{E}+}_{\lambda,10} &= \operatorname{span}\{v_j - v_{\lambda-j} \mid j \text{ odd}\}, \\ V^{\mathrm{E}+}_{\lambda,01} &= \operatorname{span}\{v_j + v_{\lambda-j} \mid j \text{ odd}\}, & V^{\mathrm{E}+}_{\lambda,11} &= \operatorname{span}\{v_j - v_{\lambda-j} \mid j \text{ even}\}. \end{aligned} \tag{4.2}$$

Denote this Γ_2-graded module by $V^{\mathrm{E}+}_\lambda$. We calculate

$$\begin{aligned} a_1(v_j \pm v_{\lambda-j}) &= \frac{i}{2}((\lambda - j + 1)(v_{j-1} \mp v_{\lambda-(j-1)}) - (j+1)(v_{j+1} \mp v_{\lambda-(j+1)})) \\ a_2(v_j \pm v_{\lambda-j}) &= -\frac{1}{2}((\lambda - j + 1)(v_{j-1} \pm v_{\lambda-(j-1)}) + (j+1)(v_{j+1} \pm v_{\lambda-(j+1)})) \\ a_3(v_j \pm v_{\lambda-j}) &= -\frac{i}{2}((\lambda - 2j)(v_j \mp v_{\lambda-j})) \end{aligned}$$

from which we can easily verify that $a_1 V^{\mathrm{E}+}_{\lambda,\gamma} \subseteq V^{\mathrm{E}+}_{\lambda,\gamma+10}$, $a_2 V^{\mathrm{E}+}_{\lambda,\gamma} \subseteq V^{\mathrm{E}+}_{\lambda,\gamma+01}$ and $a_3 V^{\mathrm{E}+}_{\lambda,\gamma} \subseteq V^{\mathrm{E}+}_{\lambda,\gamma+11}$ for each $\gamma \in \Gamma_2$. Since V^{E}_λ is irreducible, so is $V^{\mathrm{E}+}_\lambda$. By changing the H_2-parity of $V^{\mathrm{E}+}_\lambda$, we get another Γ_2-graded irreducible module $V^{\mathrm{E}-}_\lambda$, which can be obtained from $V^{\mathrm{E}+}_\lambda$ by swapping the 00- with the 11-sector and the 10- with the 01-sector. Note that $V^{\mathrm{E}+}_\lambda$ and $V^{\mathrm{E}-}_\lambda$ are not isomorphic, because $v_{\lambda/2} = (1/2)(v_{\lambda/2} + v_{\lambda-\lambda/2})$ (the unique vector with weight $\lambda/2$) will be in different sectors.

Performing similar computations with V^{O}_λ, we find two more irreducible Γ_2-graded modules $V^{\mathrm{O}+}_\lambda$ and $V^{\mathrm{O}-}_\lambda$ (these can be obtained from $V^{\mathrm{E}+}_\lambda$ by swapping the 00- with the 01-sector and the 10- with the 11-sector to get $V^{\mathrm{O}+}_\lambda$; and by swapping the 00- with the 10-sector and the 01- with the 11-sector to get $V^{\mathrm{O}-}_\lambda$). By Lemma 4.3.8 and Corollary 4.3.11 we do not need to search for any more irreducible Γ_2-graded modules in the case where λ is even.

The λ odd case

Lemma 4.4.3. *If λ is odd, then V^{E}_λ cannot be given a Γ_2-grading.*

Proof. For contradiction, assume that we can give V^{E}_λ a Γ_2-grading, $V^{\mathrm{E}}_\lambda = \bigoplus_{\gamma\in\Gamma_2} V^{\mathrm{E}}_{\lambda,\gamma}$. From the structure of V^{E}_λ, we know that $v_0 \in V^{\mathrm{E}}_{\lambda,0} = V^{\mathrm{E}}_{\lambda,00} \oplus V^{\mathrm{E}}_{\lambda,11}$. Therefore, we can write $v_0 = v_{0,00} + v_{0,11}$ for $v_{0,00} \in V^{\mathrm{E}}_{\lambda,00}$, $v_{0,11} \in V^{\mathrm{E}}_{\lambda,11}$. By Lemma 4.4.1, both $v_{0,00}$ and $v_{0,11}$ are linear combinations of v_0 and v_λ. But since $v_\lambda \in V^{\mathrm{E}}_{\lambda,1} = V^{\mathrm{E}}_{\lambda,10} \oplus V^{\mathrm{E}}_{\lambda,01}$, we must have that both $v_{0,00}$ and $v_{0,11}$ are scalar multiples of v_0. And since they sum to $v_0 \neq 0$, at least one of them must be nonzero. That is, $v_0 \in V^{\mathrm{E}}_{\lambda,00}$ or $v_0 \in V^{\mathrm{E}}_{\lambda,11}$. An analogous argument shows that $v_\lambda \in V_{\lambda,10}$ or $v_\lambda \in V_{\lambda,01}$.

If $v_0 \in V^{\mathrm{E}}_{\lambda,00}$ and $v_\lambda \in V^{\mathrm{E}}_{\lambda,01}$ then applying by a_1 we find

$$\begin{aligned} v_1 &= 2ia_1v_0 \in V^{\mathrm{E}}_{\lambda,00} \oplus V^{\mathrm{E}}_{\lambda,10} \\ v_2 &= ia_1v_1 + \frac{\lambda}{2}v_0 \in V^{\mathrm{E}}_{\lambda,00} \oplus V^{\mathrm{E}}_{\lambda,10} \\ &\vdots \\ v_{j+1} &= \frac{2i}{j+1}a_1v_j + \frac{\lambda-j+1}{j+1}v_{j-1} \in V^{\mathrm{E}}_{\lambda,00} \oplus V^{\mathrm{E}}_{\lambda,10}. \end{aligned}$$

Proceeding inductively, we find that $V^{\mathrm{E}}_\lambda \subseteq V^{\mathrm{E}}_{\lambda,00} \oplus V^{\mathrm{E}}_{\lambda,10}$. But, by applying a_1 to v_λ in a similar way, we find that $V^{\mathrm{E}}_\lambda \subseteq V^{\mathrm{E}}_{\lambda,10} \oplus V^{\mathrm{E}}_{\lambda,11}$, a contradiction! We similarly arrive at a contradiction if we assume $v_0 \in V^{\mathrm{E}}_{\lambda,11}$ and $v_\lambda \in V^{\mathrm{E}}_{\lambda,10}$; $v_0 \in V^{\mathrm{E}}_{\lambda,00}$ and $v_\lambda \in V^{\mathrm{E}}_{\lambda,10}$; or $v_0 \in V_{\lambda,11}$ and $v_\lambda \in V_{\lambda,01}$. (Note that in some of these cases we need to use a_2 instead of a_1.) □

We have just shown that V^{E}_λ cannot be given a Γ_2-grading if λ is odd. By Corollary 4.3.11, the refining extension $V^{\mathrm{E}H_2}_\lambda$ must therefore be an irreducible Γ_2-graded representation. If we take a basis $\{v_{\alpha,j} \mid \alpha \in H_2 \cong \mathbb{Z}_2,\ j = 0,\dots,\lambda\}$ for $V^{\mathrm{E}H_2}_\lambda$, where

$$\begin{aligned} V^{\mathrm{E}H_2}_{\lambda,00} &= \mathrm{span}\{v_{0,j} \mid j \text{ even}\}, \quad V^{\mathrm{E}H_2}_{\lambda,10} = \mathrm{span}\{v_{1,j} \mid j \text{ odd}\}, \\ V^{\mathrm{E}H_2}_{\lambda,01} &= \mathrm{span}\{v_{0,j} \mid j \text{ odd}\}, \quad V^{\mathrm{E}H_2}_{\lambda,11} = \mathrm{span}\{v_{1,j} \mid j \text{ even}\}, \end{aligned} \tag{4.3}$$

then we can compute the action as

$$\begin{aligned} a_1v_{\alpha,j} &= \frac{i}{2}((\lambda-j+1)v_{\alpha+1,j-1} - (j+1)v_{\alpha+1,j-1}) \\ a_2v_{\alpha,j} &= -\frac{1}{2}((\lambda-j+1)v_{\alpha,j-1} + (j+1)v_{\alpha,j-1}) \\ a_3v_{\alpha,j} &= -\frac{i}{2}(\lambda-2j)v_{\alpha+1,j}. \end{aligned}$$

By similar reasoning, we find that $V^{\mathrm{O}H_2}_\lambda$ is an irreducible Γ_2-graded representation. We can obtain $V^{\mathrm{O}H_2}_\lambda$ from $V^{\mathrm{E}H_2}_\lambda$ by swapping the 00- with the 01-sector and the 10- with the 11-sector. Note that if we change the H_2-parity of either of these modules, we get the same module. By Corollary 4.3.11, the only irreducible Γ_2-graded modules for the case where λ is odd are $V^{\mathrm{E}H_2}_\lambda$ and $V^{\mathrm{O}H_2}_\lambda$.

4.4.4 Recolouring

To recolour $\mathfrak{sl}_2$ to $\mathfrak{sl}_2^{\mathrm{c}}$ we use the multiplier $\sigma(\alpha_1\alpha_2, \beta_1\beta_2) = (-1)^{\alpha_2\beta_1}$. To obtain the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded irreducible modules for $\mathfrak{sl}_2^{\mathrm{c}}$, we use this same multiplier for the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded irreducible modules for $\mathfrak{sl}_2$.

Theorem 4.4.4. *The only finite-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded irreducible modules for $\mathfrak{sl}_2^{\mathrm{c}}$ are:*

(i) *For each even number* λ*, the four modules which are equivalent to* $(V_\lambda^{\mathrm{E}+})^\sigma$ *up to change of* $\mathbb{Z}_2 \times \mathbb{Z}_2$*-parity. The sectors of* $(V_\lambda^{\mathrm{E}+})^\sigma$ *are given as in* (4.2) *and the action is given by*

$$a_1(v_j \pm v_{\lambda-j}) = \frac{i}{2}((\lambda - j + 1)(v_{j-1} \mp v_{\lambda-(j-1)}) - (j+1)(v_{j+1} \mp v_{\lambda-(j+1)}))$$
$$a_2(v_j \pm v_{\lambda-j}) = \mp\frac{1}{2}((\lambda - j + 1)(v_{j-1} \pm v_{\lambda-(j-1)}) + (j+1)(v_{j+1} \pm v_{\lambda-(j+1)}))$$
$$a_3(v_j \pm v_{\lambda-j}) = \mp\frac{i}{2}((\lambda - 2j)(v_j \mp v_{\lambda-j})).$$

The dimension of these modules is $\lambda + 1$*.*

(ii) *For each odd number* λ*, the two modules which are equivalent to* $(V_\lambda^{\mathrm{E}H_2})^\sigma$ *up to change of* $\{00, 01\}$*-parity. The sectors of* $(V_\lambda^{\mathrm{E}H_2})^\sigma$ *are given as in* (4.3) *and the action is given by*

$$a_1 v_{\alpha,j} = \frac{i}{2}((\lambda - j + 1)v_{\alpha+1,j-1} - (j+1)v_{\alpha+1,j+1})$$
$$a_2 v_{\alpha,j} = -\frac{(-1)^\alpha}{2}((\lambda - j + 1)v_{\alpha,j-1} + (j+1)v_{\alpha,j+1})$$
$$a_3 v_{\alpha,j} = -\frac{i(-1)^\alpha}{2}(\lambda - 2j)v_{\alpha+1,j}.$$

The dimension of these modules is $2(\lambda + 1)$*.*

Proof. Use the results from the previous sections and the facts that $\sigma(10, \beta) = 1$ for all $\beta \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and

$$\sigma(01, \beta) = \sigma(11, \beta) = \begin{cases} -1 & \text{if } \beta = 10, 11 \\ 1 & \text{if } \beta = 00, 01. \end{cases}$$ □

Note that, for each $\lambda \in \mathbb{Z}_{\geq 0}$, there is a unique equivalence class of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded irreducible $\mathfrak{sl}_2^c$-modules. This corresponds exactly to the unique irreducible $\mathfrak{sl}_2$-modules, demonstrating the existence of the bijection in Theorem 4.3.10.

4.4.5 Ungraded irreducible representations for $\mathfrak{sl}_2^c$

Lemma 4.4.5. *The ungraded irreducible representations for* $\mathfrak{sl}_2^c$ *all appear as ungraded subrepresentations of the* $\mathbb{Z}_2 \times \mathbb{Z}_2$*-graded irreducible representations in Theorem 4.4.4.*

Proof. If we knew the ungraded irreducible representations for $\mathfrak{sl}_2^c$, then we could use the same process in the preceding sections to derive the same $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded irreducible modules. By applying Corollary 4.3.11 twice, these irreducible $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded modules are either irreducible as ungraded modules, a refining extension or a refining extension of a refining extension. In any case, the original ungraded irreducible modules must appear as ungraded submodules of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded modules. □

Note that $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded modules which are equivalent up to H-parity (for some $H \leq \mathbb{Z}_2$) are isomorphic as ungraded modules.

Lemma 4.4.6. *If λ is even, then $(V_\lambda^{E+})^\sigma$ is irreducible as an ungraded module.*

Proof. Choose a new basis $\{u_j \mid j = 0, \dots, \lambda\}$ where $u_j = (v_j + v_{\lambda-j}) + i(-1)^j(v_j - v_{\lambda-j})$. Note that this is not linearly dependent since $u_{\lambda-j} = (v_j + v_{\lambda-j}) - i(-1)^j(v_j - v_{\lambda-j})$. The action of $\mathfrak{sl}_2^c$ is then

$$\begin{aligned} a_1 u_j &= \frac{(-1)^{j+1}}{2}((\lambda - j + 1)u_{j-1} - (j+1)u_{j+1}) \\ a_2 u_j &= -\frac{1}{2}((\lambda - j + 1)u_{j-1} + (j+1)u_{j+1}) \\ a_3 u_j &= \frac{(-1)^{j+1}}{2}(\lambda - 2j)u_j \end{aligned} \tag{4.4}$$

with the convention that $u_{-1} = u_{\lambda+1} = 0$. In particular,

$$(a_2 + a_1)u_j = \begin{cases} -(\lambda - j + 1)u_{j-1} & \text{if } j \text{ even} \\ -(j+1)u_{j+1} & \text{if } j \text{ odd.} \end{cases}$$

$$(a_2 - a_1)u_j = \begin{cases} -(j+1)u_{j+1} & \text{if } j \text{ even} \\ -(\lambda - j + 1)u_{j-1} & \text{if } j \text{ odd.} \end{cases}$$

So, if we take an arbitrary element $\sum_j c_j u_j$ of $(V_\lambda^{E+})^\sigma$, then we can alternate applying $(a_2 + a_1)$ and $(a_2 - a_1)$ (i.e. we apply $((a_2 - a_1)(a_2 + a_1))^k$ or $(a_2 - a_1)((a_2 + a_1)(a_2 - a_1))^k$ for some non-negative integer k) to raise/lower the basis elements of $\sum_j c_j u_j$. Eventually, this procedure will yield a scalar multiple of either u_0 or u_λ, from which we can generate the entire module. Since the entire module can be generated from a single vector, $(V_\lambda^{E+})^\sigma$ is irreducible. □

Now we consider the case when λ is odd. Then $(V_\lambda^{EH_2})^\sigma = U_\lambda^{++} \oplus U_\lambda^{+-} \oplus U_\lambda^{-+} \oplus U_\lambda^{--}$ where

$$\begin{aligned} U_\lambda^{\zeta\xi} &= \operatorname{span}\{u_j^{\zeta\xi} \mid j = 0, 1, \dots, (\lambda-1)/2\} \\ u_j^{\zeta\xi} &= v_{0,j} + \zeta i(-1)^j v_{1,j} + \xi v_{0,\lambda-j} - \zeta\xi i(-1)^j v_{1,\lambda-j} \end{aligned}$$

for $\zeta, \xi \in \{-1, +1\}$. The action of $\mathfrak{sl}_2^c$ is given by

$$\begin{aligned} a_1 u_j^{\zeta\xi} &= -\frac{\zeta}{2}(-1)^j((\lambda - j + 1)u_{j-1}^{\zeta\xi} - (j+1)u_{j+1}^{\zeta\xi}) && \left(j < \frac{\lambda-1}{2}\right) \\ a_1 u_{(\lambda-1)/2}^{\zeta\xi} &= -\frac{\zeta}{2}(-1)^{(\lambda-1)/2}\left(\frac{\lambda+3}{2}u_{(\lambda-3)/2}^{\zeta\xi} - \xi\frac{\lambda+1}{2}u_{(\lambda-1)/2}^{\zeta\xi}\right) \\ a_2 u_j^{\zeta\xi} &= -\frac{1}{2}((\lambda - j + 1)u_{j-1}^{\zeta\xi} + (j+1)u_{j+1}^{\zeta\xi}) && \left(j < \frac{\lambda-1}{2}\right) \\ a_2 u_{(\lambda-1)/2}^{\zeta\xi} &= -\frac{1}{2}\left(\frac{\lambda+3}{2}u_{(\lambda-3)/2}^{\zeta\xi} + \xi\frac{\lambda+1}{2}u_{(\lambda-1)/2}^{\zeta\xi}\right) \\ a_3 u_j^{\zeta\xi} &= -\frac{\zeta}{2}(-1)^j(\lambda - 2j)u_j^{\zeta\xi} \end{aligned} \tag{4.5}$$

with the convention $u_{-1} = 1$. (Note that $u_j^{\zeta\xi}$ was chosen so that it was an eigenvector of a_3.) Clearly $U_\lambda^{\zeta\xi}$ gives rise to four non-isomorphic modules, since the action of a_1, a_2 and a_3 on $u_{(\lambda-1)/2}^{\zeta\xi}$ (the unique eigenvector for a_3 which has eigenvalue with absolute value $1/2$) is different for different values of ζ, ξ.

Lemma 4.4.7. *$U_\lambda^{\zeta\xi}$ is irreducible as an ungraded module.*

Proof. Observe that, for $j < (\lambda-1)/2$,

$$(a_2+a_1)u_j^{\zeta\xi} = \begin{cases} -(\lambda-j+1)u_{j-1}^{\zeta\xi} & \text{if } \zeta(-1)^j = 1 \\ -(j+1)u_{j+1}^{\zeta\xi} & \text{if } \zeta(-1)^j = -1 \end{cases}$$

$$(a_2+a_1)u_{(\lambda-1)/2}^{\zeta\xi} = \begin{cases} -\frac{\lambda+3}{2}u_{(\lambda-3)/2}^{\zeta\xi} & \text{if } \zeta(-1)^{(\lambda-1)/2} = 1 \\ -\xi\frac{\lambda+1}{2}u_{(\lambda-1)/2}^{\zeta\xi} & \text{if } \zeta(-1)^{(\lambda-1)/2} = -1 \end{cases}$$

$$(a_2-a_1)u_j^{\zeta\xi} = \begin{cases} -(j+1)u_{j+1}^{\zeta\xi} & \text{if } \zeta(-1)^j = 1 \\ -(\lambda-j+1)u_{j-1}^{\zeta\xi} & \text{if } \zeta(-1)^j = -1 \end{cases}$$

$$(a_2-a_1)u_{(\lambda-1)/2}^{\zeta\xi} = \begin{cases} -\xi\frac{\lambda+1}{2}u_{(\lambda-1)/2}^{\zeta\xi} & \text{if } \zeta(-1)^{(\lambda-1)/2} = 1 \\ -\frac{\lambda+3}{2}u_{(\lambda-3)/2}^{\zeta\xi} & \text{if } \zeta(-1)^{(\lambda-1)/2} = -1 \end{cases}$$

Take an arbitrary element $\sum_j c_j u_j$ of $U_\lambda^{\zeta\xi}$. Similar to the proof of Lemma 4.4.6 we alternate applying a_2+a_1 and a_2-a_1. In this case, we are able to obtain a scalar multiple of u_0 from $\sum_j c_j u_j$, from which we can generate the entire module. □

In summary, we have proven the following theorem:

Theorem 4.4.8. *The only finite-dimensional irreducible ungraded modules for $\mathfrak{sl}_2^c$ are*

(i) For each even number λ, the module $(V_\lambda^{\mathrm{E}+})^\sigma$ with action given in (4.4). The dimension of these modules is $\lambda+1$.

(ii) For each odd number λ and each $\zeta,\xi \in \{-1,+1\}$, the module $U_\lambda^{\zeta\xi}$ with action given in (4.5). The dimension of these modules is $(\lambda+1)/2$.

Remark 4.4.9. It is clear that the modules $U_\lambda^{+\xi}$ and $U_\lambda^{-\xi}$ are Γ/H-representative negations of each other, where $\Gamma = \mathbb{Z}_2\times\mathbb{Z}_2/\{00,01\}$ and $H=\Gamma$ (with representatives $\{[00]\}$). If we instead choose $\Gamma = \mathbb{Z}_2\times\mathbb{Z}_2/\{00,11\}$, $H=\Gamma$, then we find that the Γ/H-representative negation of $U_\lambda^{\zeta+}$ (with representatives $\{[00]\}$) is isomorphic to $U_\lambda^{\zeta-}$, with isomorphism given by $u_j^{\zeta+}\mapsto(-1)^j u_j^{\zeta-}$.

The above remark shows that $U^{\zeta\xi}$ are all equivalent for different values of ζ,ξ. Consequently, there is a unique equivalence class in Theorem 4.4.8 for each $\lambda\in\mathbb{Z}_{\geq 0}$. This corresponds exactly to the equivalence classes for $\mathbb{Z}_2\times\mathbb{Z}_2$-graded $\mathfrak{sl}_2^c$-modules in Theorem 4.4.4, again demonstrating the existence of the bijection in Theorem 4.3.10.

Remark 4.4.10. Although the ungraded $\mathfrak{sl}_2^c$ representations were derived from the ungraded $\mathfrak{sl}_2$ representations, they have a remarkably different structure. This shows that, despite the procedure outlined in this chapter, colour algebras still have an interesting representation theory.

Chapter 5

Conclusion

Throughout this thesis we have examined physical applications and mathematical structures of Lie colour algebras.

Discolouration is a powerful tool that allowed us, in Chapter 3, to classify the two Lie colour algebra symmetries of the Lévy-Leblond equation that had previously been discovered in [4]. We found that one algebra is isomorphic to $\mathfrak{osp}(1,0|2,0) \oplus \mathfrak{osp}(1,0|0,2)$ and the other is isomorphic to $\mathfrak{osp}(1,0|0,2) \oplus \mathfrak{osp}(1,1|2,0)$. Discolouration allowed us to apply pre-existing classification results about Lie superalgebras to these Lie colour algebras, and this classification would have been more difficult otherwise. Due to its ability to carry across existing results for Lie superalgebras, we hope that use of discolouration becomes more widespread in the literature.

In Chapter 3, we also discovered a new $\mathbb{Z}_2^3$-graded Lie colour algebra symmetry for a simple version of the Lévy-Leblond equation and showed that this Lie colour algebra could be used to aid in solving this version of the equation. This further demonstrates the potential utility of Lie colour algebras in physical applications. Moreover, this new discovery indicates that there may be other useful Lie colour algebras for the Lévy-Leblond equation that are waiting to be discovered.

It is natural to ask whether we can find a Lie colour algebra in a more complicated version of the equation. For instance, we could consider the Lévy-Leblond equation in $(1+d)$-dimensions, rather than just $(1+1)$-dimensions. This would be more difficult, since the construction of the $\mathbb{Z}_2^3$-graded algebra relied on the fact that the differential operator ∂_1 in the time-independent equation could be expressed in terms of gamma matrices (acting on a single eigenspace). In $(1+d)$-dimensions for $d \geq 2$ there will be more than one differential operator, and it is unlikely that we would be able to solve for all differential operators in general.

We could also investigate whether a Lie colour algebra appears after adding a potential to the equation. There has been some work in this direction: in [4], the authors showed that the Schrödinger symmetry algebra has no $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded extension in the presence of a quadratic potential. This is a somewhat disappointing negative result, though there could be a Lie colour algebra for this equation which is not related to the Schrödinger algebra.

The Lévy-Leblond equation is one of the few pre-existing physical equations where Lie colour algebras have appeared. It is a challenging open problem to find Lie colour algebras in other physical systems. Given its close relationship with the Lévy-Leblond equation, the Dirac equation is an obvious place to look for such Lie colour algebra symmetries.

In Chapter 4 we studied representation theory, which is often a useful tool when studying physical systems. It is desirable to have the ability to classify irreducible representations (see e.g. [7]). We showed that there is a correspondence between the irreducible representations (both graded and ungraded) of Lie colour algebras and the irreducible representations of Lie superalgebras. We showed how this correspondence could be strengthened to a bijection for $\mathbb{Z}_2^n$-graded algebras, and gave an example of applying this correspondence to $\mathfrak{sl}_2^c$.

Unfortunately, just like for Lie superalgebras, many finite-dimensional Lie colour algebra representations will not be completely reducible. With this in mind, it would be beneficial to have a similar correspondence between indecomposable representations, not just irreducible ones. Another natural next step is to apply the correspondence to known classification results for Lie superalgebra representations. This would allow us to build up a classification of irreducible Lie colour algebra representations.

In this thesis, we have shown how Lie colour algebras can be applied to a physical system; namely, via symmetry algebras of the Lévy-Leblond equation. We have used and developed mathematical techniques related to Lie colour algebras with the hope that they will aid in such physical applications.

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Appendix A

Appendix

A.1 Gradings of Clifford algebras

The relations (3.12) naturally give rise to a complete set of both commutation relations and anticommutation relations:

$$\begin{aligned} &[I,\gamma_\pm] = \left[I,\gamma^1\right] = 0 && \left[\gamma^1,\gamma_\pm\right] = \pm 2\gamma_\pm && [\gamma_+,\gamma_-] = \gamma^1 \\ &\{I,\gamma_\pm\} = 2\gamma_\pm && \left\{\gamma^1,\gamma_\pm\right\} = 0 && \{\gamma_+,\gamma_-\} = I \\ &\left\{I,\gamma^1\right\} = 2\gamma^1 && \left\{\gamma^1,\gamma^1\right\} = 2I && \{\gamma_\pm,\gamma_\pm\} = 0. \end{aligned} \tag{A.1}$$

That the gamma matrices close under commutation and anticommutation relations is one condition we need for a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded structure. The other condition we need is a grading $\bigoplus_{\xi\in\mathbb{Z}_2\times\mathbb{Z}_2} \mathfrak{g}_\xi$ which satisfies $[\![\mathfrak{g}_\alpha, \mathfrak{g}_\beta]\!] \subseteq \mathfrak{g}_{\alpha+\beta}$. We have two options for a non-trivial $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading:

$$\begin{aligned} &\mathfrak{g}_{00} = \mathrm{span}\{I\} && \mathfrak{g}_{01} = \mathrm{span}\{\gamma_+,\gamma_-\} && \mathfrak{g}_{10} = \{0\} && \mathfrak{g}_{11} = \mathrm{span}\{\gamma^1\} \\ &\mathfrak{g}_{00} = \mathrm{span}\{I\} && \mathfrak{g}_{01} = \mathrm{span}\{\gamma_+\} && \mathfrak{g}_{10} = \mathrm{span}\{\gamma_-\} && \mathfrak{g}_{11} = \mathrm{span}\{\gamma^1\}. \end{aligned}$$

In addition to the above gradings, there are potentially more interesting ones where the basis of each sector is a linear combination of generators. That the gamma matrices can be given a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded structure gives a partial explanation for the appearance of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded symmetry algebras. However, the above gradings cannot be used to determine the gradings of the symmetry algebras.

The appearance of both commutation relations and anticommutation relations does not only occur in $(1+1)$-dimensions. The gamma matrices in (3.8) are realised in the complexification of the Clifford algebra $C\ell_{1,d}(\mathbb{R})$ (namely $C\ell_{1,d}(\mathbb{R}) \otimes \mathbb{C}$). Note that $C\ell_{1,d}(\mathbb{R})$ is finite-dimensional, of dimension 2^{1+d} [62, Chapter 14]. Since $C\ell_{1,d}(\mathbb{R})$ is an associative algebra, it can be given the structure of a Lie algebra with commutator Lie bracket. Thus, $C\ell_{1,d}(\mathbb{R})$ can be described in terms of both commutation and anticommutation relations. If we could find a suitable grading, we could make $C\ell_{1,d}(\mathbb{R})$ a Lie colour algebra. We showed this was possible for

$C\ell_{1,1}(\mathbb{R}) = \text{span}_{\mathbb{R}}\{I, \gamma_+, \gamma_-, \gamma^1\}$ above, and this is always possible [63]. (However, in [63] γ_+ and γ_- are not homogeneous elements which may be desired in the context of the Lévy–Leblond equation.)

In addition, the tensor product of a Clifford algebra with a non-colour algebra can often give rise to a colour algebra [71]. The above discussion perhaps gives a partial explanation for the presence of a colour symmetry algebra for the Lévy-Leblond equation.

The $(1+1)$-dimensional case is unique because the (anti)commutation relations (A.1) contain only the gamma matrices which appear in the corresponding Lévy-Leblond equation. The reason for this is that these gamma matrices span the Clifford algebra $C\ell_{1,1}(\mathbb{R})$. This is not true for higher dimensions. One could choose a small representation to minimise the number of distinct gamma matrices needed (and add more relations), but the choice of this representation would be somewhat arbitrary.